TITLE **The Existence And Stability Of Fractional Quantum Hall States**

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# The Existence And Stability Of Fractional Quantum Hall States

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**Abstract**

I examine the existence and stability of spin-polarised fractional quantum Hall (FQH) states. I show that the generalisations of the Laughlin state to higher ($n > 0$) Landau levels are not always stable. I show that an instability arises because the electron-electron interaction which would give rise to a Laughlin-like ground state at a filling fraction $\nu_n$ of the $n^{th}$ Landau level with $\nu_n^{-1} \leq 2n + 1$ is physically unrealisable. I argue that this destabilization of FQH states emerges naturally in the cooperative ring exchange theory of the FQHE. I claim that a generalised cooperative ring exchange theory predicts the occurence of two phase boundaries. The first boundary is between a Wigner-crystal and Laughlin-like ground state at small filling fractions. The second corresponds to the boundary a Laughlin-like ground state and a compressible fluid ground state. This second transition occurs only in higher Landau levels for a spin polarised electron gas.

I suggest that a hierarchical picture of the FQHE in which a new ground state results from the condensation of a quasiparticle gas from a preceeding level should not be taken too literally. I argue that the regime $5 > \nu_1^{-1} > 3$ in the first ($n = 1$) Landau level is approximately analogous to the regime $3 > \nu_0^{-1} > 1$ in the lowest Landau level independent of whether the ground state is Laughlin-like at $\nu_1 = \frac{1}{3}$ or not. This claim is supported by results of numerical studies for electrons in a partially filled first Landau level.

I study the dispersion of collective excitations in the FQHE using Feynman-Bijl ansatz wavefunctions. I take the pair correlation functions which forms the basis for the variational estimate for the dispersion, from the direct diagonalisations of the Hamiltonian for a small number of electrons confined to the surface of a sphere. I found that the dispersion of collective excitations for the $\nu_1 = \frac{4}{3}$ state are characteristic of a FQH state, with a well defined gap and roton minimum. This contrasts with the dispersion of collective excitations for the 'parent state' at $\nu_1 = \frac{1}{3}$ which has only a small gap and no clear roton minimum.

## Contents

**List of Illustrations**

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**Declaration**

This thesis contains an account of my own independent research work performed in the Department of Physics at the University of Warwick between October 1986 and May 1989 under the general supervision of Dr. N.d'Ambrumenil.

Some of the work has previously been published in:-

1. Fractional quantum Hall states in higher Landau levels.
   N. d'Ambrumenil and A.M.Reynolds
   J.Phys.C. 21 119-132, 1988

2. Collective excitations in the Fractional quantum Hall effect
   A.M.Reynolds and N. d'Ambrumenil
   J.Phys.C. 21 5643-5651, 1988

3. A generalised cooperative ring exchange theory of the fractional quantum Hall effect.
   A.M.Reynolds and N.d'Ambrumenil
   In preparation

# Chapter 1

# The Fractional Quantum Hall Effect

## 1.1 Introduction

One of the most remarkable magnetotransport phenonema to be discovered is the fractional quantum Hall effect or FQHE. The FQHE refers to the observation of plateaux in the Hall resistance of some semiconductor structures at rational multiples $\nu^{-1}$ of $\frac{h}{e^2}$ and the related dramatic reduction (upto 10 orders of magnitude) in the longitudinal resistance. The fractional quantisation can be highly accurate. For plateaux at $\nu = \frac{1}{3}$ and $\nu = \frac{2}{3}$, the accuracy is one part in a million in the vicinity of the centre of the Hall plateaux. The most pronounced plateau is at $\nu = \frac{1}{3}$ (Tsui et al, 1982) followed in approximate order by $\frac{2}{3}$, $\frac{2}{5}$, and $\frac{3}{5}$ ... (Chang et al 1984). Fig.1

shows some of the recent observations of the FQHE (Willett et al, 1988).

The plateaux tend to occur in semiconductor structures with very high mobilities ($> 10^5 cm^2/Vsec$) at high perpendicular magnetic fields ($B > 5T$), and at low temperatures ($T < 2K$). The experiments are usually performed in a geometry similar to that shown in Fig.2.

In what follows, I introduce aspects of the current theoretical understanding of the FQHE which are of direct relevance to the body of this thesis. In the next section I discuss some properties of electrons in the semiconductor structures in which the FQHE is observed. I then introduce the most generally accepted theory of the FQHE, due to Laughlin 1983. I do this in three steps. I begin by describing Laughlin's Gedankenexperiment in which the quantisation of Hall conductance is shown to arise in any model, in which there is a gap in a band of delocalised states and dissipationless flow. In section 1.4 I show how Laughlin's Gedankenexperiment can be invoked to form the basis of a theory for the FQHE. Then in section 1.5 I show how a many-body wavefunction introduced by Laughlin incorperates the two properties necessary for quantization of the Hall conductance. I demonstrate that Laughlin's wavefunction describes an incompressible fluid separated by a gap from fractionally charged excitations. In section 1.6 I discuss the collective excitations in the FQHE. In section 1.7 I show how observations of the FQHE at fractions $\nu = \frac{p}{q}$ with $p \neq 1$ might be attributed to a hierarchy of Laughlin-like condensates, in which the fractionally charged excitations of one level in the hierarchy are thought to form

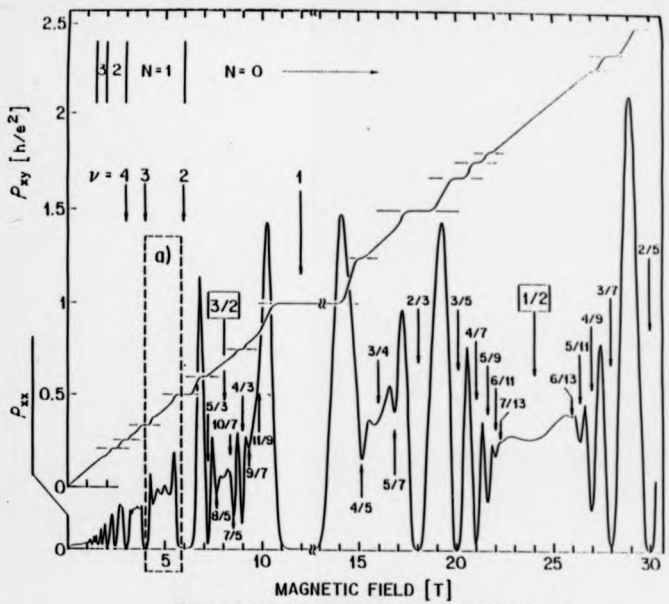


**Figure 1. Overview of the diagonal resistivity $\rho_{xx}$ and the Hall resistance $\rho_{xy}$ observed in a GaAs/GaAlAs heterojunction by Willett et al 1988, as a function of magnetic field strength. Plateaux observed in $\rho_{xy}$ along with minima in $\rho_{xx}$ indicate fractional quantum Hall states. Filling fractions $\nu$ and Landau level indices are indicated.**

the particles/holes of the next level in the hierarchy. In section 1.8 I introduce the cooperative ring exchange theory of the FQHE. The central theme of this thesis, namely the existence and stability of FQH states is presented in section 1.9. Section 1.10 is a summary. The FQHE has been extensively reviewed elsewhere (see e.g. Girvin and Prange 1987).

## 1.2 Some Preliminaries

In this section I discuss some properties of electrons in the semiconductor structures in which the FQHE is observed. These properties are of relevance to an understanding of the FQHE. At low temperatures ($T < 4K$) the inversion layer formed in metal oxide semiconductor field-effect transistors (MOSFETs) and $GaAs/GaAl_xAs_{1-x}$ heterojunctions is well described as a two dimensional electronic system. This is because the energy level spacing for motion perpendicular to the plane of the structure is large compared for example, to thermal energies or the Fermi energy. In the plane of the structure electrons move like quasi-free particles with their interactions being dominated by electron-electron repulsion.

In a magnetic field perpendicular to the plane of motion and in the absence of impurities or an electric field, the kinetic energy spectrum of these two dimensional electrons becomes quantised into Landau levels. The kinetic energy spectrum of the electrons is given by

$$E(n) = \left(n + \frac{1}{2}\right)\hbar\omega_c \tag{1.1}$$

where $\omega_c = eB/m^*$ is the cyclotron frequency of an electron with effective mass $m^*$ and n is the Landau level index.

In each Landau level there are $eB/h$ degenerate states (cyclotron orbits) per unit area i.e. one per flux quantum in each Landau level (see Appendix A). The ratio of the number of electrons in a Landau level to the number of single particle cyclotron orbits (flux quanta) is called the filling fraction and is usually denoted by $\nu$. The filling fraction is found to play a crucial role in determining the state of a two dimensional electron system. Only when the filling fraction is rational can electrons condense into a FQH state.

When impurities are included, these may shift the energies of the cyclotron orbits and give rise to some localised states centred on individual scattering centres. The Landau levels are expected to spread out into bands so that the density of states may look like that shown in Fig.3. Close to the original energies $\left(n + \frac{1}{2}\right)\hbar\omega_c$ states are generally believed to remain delocalised, while localised states are thought to exist in the tails of the bands. Only delocalised states can carry current, so that between the upper tail of one band and the lower tail of the next band there will be a 'mobility gap'. If there is indeed a mobility gap and if the Fermi energy lies in this gap, then an elegant argument due to Laughlin shows that the Hall conductance must be quantised. In this particular case the Hall conductance becomes quantised at integer multiples of $\frac{e^2}{h}$.

In what follows I discuss Laughlin argument and show how it can account

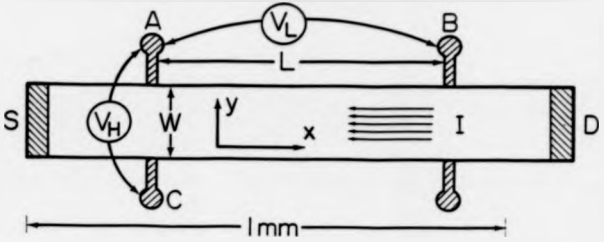


**Figure 2.** A standard sample configuration for the measurement of diagonal resistivity $\rho_{xx}$ and the Hall resistance $\rho_{xy}$.

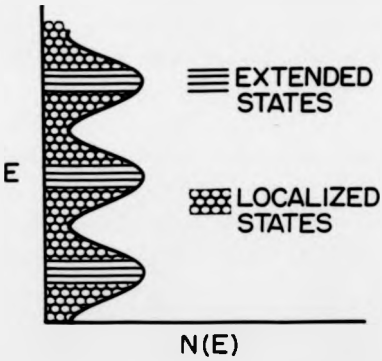


**Figure 3.** Possible density of electronic states in a perpendicular magnetic field for system of quasi two-dimensional electrons containing impurities.

for the FQHE where the 'mobility gap' is presumably caused by electron-electron interactions.

## 1.3 Laughlin's Gedankenexperiment

The universality of the FQHE suggested an explaination based on a fundamental priniciple rather than a microscopic transport theory. This fundamental prinicipal was first stated by Laughlin, 1981. Laughlin deduced from the implications of a Gedankenexperiment, outlined below, that the Hall conductance of a system of charged particles must be quantised in units of $\frac{e^2}{h}$, if the Fermi energy lies in a mobility gap.

Laughlin considered a system of electrons confined to the surface of a cylinder, with a homogeneous magnetic field B perpendicular to the cylinder surface at all points. Laughlin introduced a homogeneous flux $\Phi_x$, threading the circuit of the cylinder but which vanishes at the cylinder surface (i.e. in the 2D system under consideration), see Fig.4.

The application of a transverse voltage $V_x$, induces a current flow around the cylinder.

Laughlin imagined adiabatically changing the flux $\Phi_x$ by one flux quantum $\Delta\Phi_x = \frac{h}{e}$. This changes the vector potential for the electrons on the cylinder

$$A \rightarrow A + \frac{h}{e}y/L$$

corresponding to the gauge change

$$\Psi \rightarrow \exp(-2\pi i y/L)\,\Psi$$

The physical properties of the system must of course be unaffected by this gauge change. In particular when the Fermi energy of the system lies within a band of localised states the occupied bands of delocalised states before the gauge change must be occupied afterwards. While the occupation numbers of the localised states cannot change. Only the occupation numbers of edge states can change, corresponding to the transfer of particles from one edge of the cylinder to the other.

This transfer of particles is readily seen in the Landau gauge where delocalised states of the form (see Appendix A)

$$|\, n,k >\sim \exp(iky)\, H_n\left(x - k\left(\frac{\eta}{e}\right)\right) \exp\left(-\left(x - k\left(\frac{\eta}{e}\right)\right)^2/2\right) \qquad (1.2)$$

respond to the gauge change by shifting their centres by an amount $\Delta\phi/\phi_0$

$$k \rightarrow k + \Delta\phi/\phi_0$$

so that each state maps into its neighbour. The net result of this is the transfer of exactly one state from one edge to the other.

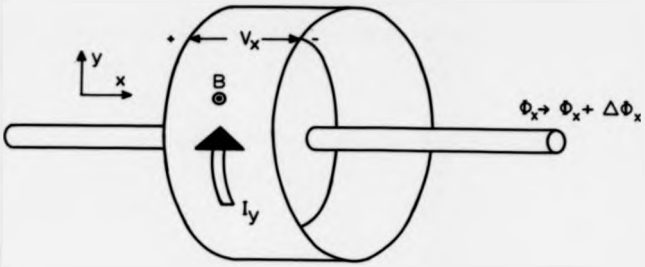


**Figure 4.** Laughlin's Gedankenexperiment. By considering the gauge change $\Phi_x \rightarrow \Phi_x + \Delta\Phi_x$ one can deduce that when the Fermi energy lies a mobility gap, $\rho_{xy} = V_x/I_y$ takes on only quantized values.

Suppose n particles each having charge q are transferred from one edge of the cylinder to the other during the gauge change, then the potential energy of the system is increased by $\Delta U$

$$\Delta U = nqV_x \tag{1.3}$$

As the change occurred adiabatically this energy must compensate exactly the work done by the system, which by Faraday's law is $\Delta W$, with

$$\Delta W = -\int dt I_y \left(\frac{d\phi_x}{dt}\right) = -I_y \left(\frac{h}{e}\right) \tag{1.4}$$

So

$$0 = \Delta U + \Delta W = nqV_x - I_y \left(\frac{h}{e}\right) \tag{1.5}$$

or

$$\frac{I_y}{V_x} = -\sigma_{xy} = \frac{nqe}{h} \tag{1.6}$$

The Hall conductance thus measures the number of charge carriers transferred in the Gedankenexperiment. The quantization of Hall conductance is valid irrespective of the form of the many body Hamiltonian. That is any model with a gap in the band of delocalized states and dissipationless flow must give quantization of the Hall conductance.

## 1.4 Many Body Gap and Fractional Charge

Laughlin's Gedankenexperiment can be invoked to form the basis for a theory of the FQHE . The observation of plateaux in the Hall conductance would then imply the

existence of a mobility gap in the system. This gap would presumably be caused by electron - electron interactions as the single particle states, the cyclotron orbits are degenerate. Then when the Fermi-level is in this gap, the Hall conductivity is according to the Gedankenexperiment given by

$$| \sigma_{xy} | = \frac{nqc}{h} \tag{1.7}$$

Comparing this, with the experimentally observed value of $\frac{1}{3}\frac{e^2}{h}$ at $\nu = \frac{1}{3}$ say leads to

$$n \mid \frac{q}{e} \mid = \frac{1}{3} \tag{1.8}$$

This result implies the existence of fractionally charged particles, with charge $| q | = \frac{e}{3}$. Laughlin 1983 developed this idea and constructed a trial ground state wavefunction, whose excitations do at least appear to be fractionally charged objects.

An alternative view may be argued as follows. The Gedankenexperiment presupposes that the bulk occupation numbers cannot change under the gauge transformation. If however there were more than one equivalent ground state then the gauge transformation might map electrons not into the same ground state but into one of the equivalent ground states. In the Landau gauge with the application of periodic boundary conditions, at a filling $\nu = \frac{p}{q}$ there is indeed a q-fold 'centre of mass' degeneracy in the ground state (Thouless 1985 ). Only changes of flux by multiples of q ($\frac{h}{e}$) map these systems into themselves again with a corresponding change of one electron/hole in the edge states. The measurement of hall conductance can be related to this centre of mass degeneracy (Niu et al 1985).

## 1.5 Laughlin's Wavefunction

In this section I introduce Laughlin's wavefunction and discuss Laughlin's claim that close to certain filling fractions of the lowest Landau level, two-dimensional electrons in a perpendicular magnetic field condense into a new phase of matter, the Laughlin fluid, described by Laughlin's wavefunction. The Laughlin fluid is incompressible and separated by a gap from fractionally charged excitations. The fractional charge of these excitations is essentially what is measured by the Hall resistance.

Laughlin 1983, constructed a trial ground state wavefunction

$$\Psi_L(r_1, \ldots r_n) = \prod_{i<j}^{N} (z_i - z_j)^m \exp - \sum_i |z_i|^2/4 \qquad (1.9)$$

where $z_i = x_i + i\left(\frac{q}{|q|}\right) y_i$, is the coordinate of the $i^{th}$ particle, in units of the magnetic length $l_0 = (eB/\hbar)^{\frac{1}{2}}$, which I set equal to 1 for convenience. Here q is the electronic charge carried by the particle. For a spin polarised system of electrons m must be an odd integer because $\Psi_L$ must be antisymmetric under exchange of any pair of electrons.

Since all terms in the Laughlin state are of the form

$$\prod_i (z_i)^p \exp\left[-\frac{|z_i|^2}{2}\right] \qquad (1.10)$$

it is automatically made up only of cyclotron orbits in the lowest Landau level (Appendix B).

The Laughlin state $\Psi_L$ describes a fluid. This can be seen by writing $|\Psi_L|^2$ as

a classical distribution function

$$|\Psi_L|^2 \sim \exp -\beta\phi$$

$$\text{with} \quad \beta = 1/m$$

$$\text{and} \quad \phi = -\sum_{i<j} 2m^2 \log|z_i - z_j| + \frac{1}{2}m\sum_i |z_i|^2 \qquad (1.11)$$

Here $\phi$ is the potential of a classical two dimensional one component plasma or 2DOCP (Caillol et al 1982). The classical 2DOCP describes a system of identical particles carrying charge m, interacting through a logarithmic potential, embedded in a uniform neutralising background charge. This system as been well studied and for $2m < 140$, $\phi$ is known to describe a fluid (Caillol et al 1982) with density $m^{-1}$. This fluid is incompressible as the length scale $l_0$, cannot change without introducing or removing flux quanta.

Laughlin calculated the total energy per particle for a system of electrons described by his trial wavefunction and found for densities corresponding to observed FQH states, that it is lower than the possible alternative charge density wave and Wigner crystal ground states. The validity of the Laughlin state is well supported by the results of numerical studies of small systems. Where large overlaps, of order unity, between the Laughlin state and the exact ground state, for systems of electrons interacting through the Coulomb interaction have been reported (e.g. see d'Ambrumenil and Reynolds 1988).

In order to explain the FQHE it was necessary to show how current can flow without dissipation. This is presumably related to the existence of a gap in the

excitation spectrum of the system. Laughlin imagined generating elementary excitations 'quasiparticles' and 'quasiholes' from $\Psi_L$ by adding or removing flux quanta at some point $z_0 = x_0 + i\frac{q}{|q|}y_0$. Laughlin suggested trial wavefunctions

$$\Psi_L^+ = \prod_k (z_k - z_0)\,\Psi_L \qquad \Psi_L^- = \prod_k \left(2\frac{\delta}{\delta z_k} - z_0\right)\Psi_L \tag{1.12}$$

for the quasiholes and quasiparticles.

$|\,\Psi_L^+\,|^2$ describes the same classical plasma as before, except for an additional particle, a quasihole, with charge +1 located at $z_0$. The plasma will completely screen the quasihole so that there is an accumulation of an equal but opposite amount of charge near $z_0$. In the electron system the electrons that screen out the charge at $z_0$ have charge $-1$ rather than $-m$, the charge carried by particles in the 2DOCP analog of the Laughlin state. There will therefore be an absence of $\frac{1}{m}^{th}$ of an electron in the neighbourhood of $z_0$, and hence a local excitation with charge $+\frac{e}{m}$. Although there is no direct way of mapping quasiparticles onto the 2DOCP, a similar intrepretation may be applied to $|\,\Psi_L^-\,|^2$ (Laughlin 1984). There is thus an electrostatic energy associated with the quasiparticle and quasihole, so that the energy to produce a quasihole - quasiparticle pair is finite. The Laughlin ground state is therefore separated by a gap from fractionaly charged excitations. At low temperatures this gap will presumably result in dissipativeless flow.

The quasiholes and quasiparticles are both charged and bind a magnetic flux quantum. When two such objects exchange positions anomalous phase changes can be produced by the Aharonov Bohm effect. The quasiholes and quasiparticles are

found to obey fractional statistics (Arovas et al 1984).

## 1.6 Collective Excitations

The low lying excitations from the Laughlin state can be thought of as quasiexcitons, bound pairs of Laughlin quasiholes and quasiparticle, which at small spatial separations look like density wave excitations. Girvin et al 1986 (GMP) suggested using an adapted Feynman-Bijl ansatz wavefunction to describe these excitations. Making the approximation that only one 'band' of such collective excitations coupled to the ground state they were able to predict the dispersion of this band variationally.

GMP considered the trial wavefunction formed by operating on the homogeneous ground state with the density operator projected onto the lowest Landau level. They were then able to relate the dispersion of the collective excitations to the pair correlation function, took from results obtained for the 2DOCP.

The method of GMP gives a numerical estimate for the dispersion of collective excitations, which is in good agreement with the exact numerical results for small systems (Haldane and Rezayi, 1985).

## 1.7 A Hierarchy of Quasiparticles and Quasiholes

The Laughlin state for a system of spin polarised electrons can only be constructed for filling fractions $\nu = \frac{1}{m}$ with m an odd integer. Observations of the FQHE at $\nu = \frac{2}{3}, \frac{3}{5}, \ldots$ have been attributed to a hierarchy of quasiparticle or quasihole

condensates (Haldane 1983, Halperin 1983). At each level of the hierachy a new ground state is thought to result from the condensation of a quasiparticle gas from the preceding level. For example as the magnetic field decreases from its value at $\nu = \frac{1}{3}$, fewer electrons can be accomodated because the number of cyiotron orbits decreases with magnetic field, quasiparticles are nucleated which at a filling $\nu = \frac{2}{5}$ are believed to condense to form a 'higher order Laughlin fluid', the $\nu = \frac{2}{5}$ ground state. With a further reduction in magnetic field the quasiparticles of the $\nu = \frac{2}{5}$ ground state condense to form a new ground state at $\nu = \frac{3}{7}$. The hierarchy of Laughlin-like condensates continues in a similar manner. If the magnetic field is increased from its value at $\nu = \frac{1}{3}$, quasiholes are formed which condense at $\nu = \frac{2}{7}$ and so on.

I now outline a more rigorous discussion of the hierarchy due to Haldane 1983. In Haldane's approach quasiparticles and quasiholes are treated as Bosons. Other versions of the hierarchy scheme, due to Laughlin 1984 and Halperin 1984 differ in their assignment of statistics to the quasiparticles and quasiholes. This difference can be traced back to their different definitions of relative angular momentum for pairs of quasiparticles or quasiholes. If the parent state is a spin polarised Laughlin state with filling fraction $\nu = \frac{1}{m}$, then $N_\phi = mN_e$ where $N_\phi$ is the number of independent single electron states (flux quanta) and $N_e$ is the number of electrons. The equations at the first level hierarchy are

$$N_\phi = mN + \alpha N_{qp} \tag{1.13}$$

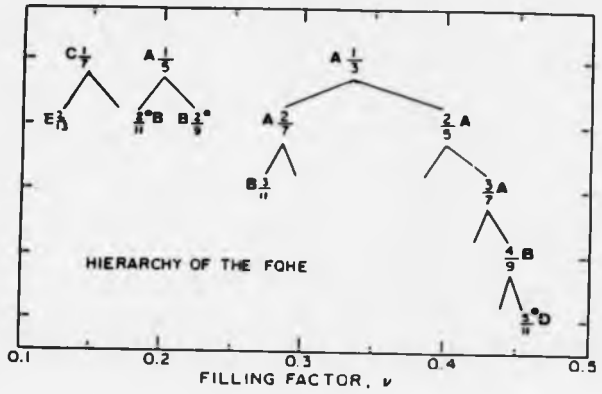


**Figure 5.** The hierarchy of Laughlin fluids. Experimentally observed fractional quantum Hall states are indicated.

- **A** Chang et al, 1984
- **B** Mallett et al, 1988
- **C** Goldman et al, 1988
- **D** Willett et al, 1987
- **E** suggested by R.Clarke at LDS Summer School, 1988.

• - indicates weak minima in $\rho_{xx}$ seen, with no plateau structure in $\rho_{xy}$.

$$N = pN_{qp} \tag{1.14}$$

In equation 1.13 $N_{qp}$ is the number of quasiparticles ($\alpha = -1$) or quasiholes ($\alpha = 1$). Equation 1.14 is analogous to the one for parent Laughlin state, but with $N_\phi$ the number of independent electron states replaced by $N$ the number of independent quasiparticle states. p is even because of the nomial Bose statistics of the quasiparticles. The filling fraction for a fluid at the first level of the hierarchy is

$$\nu = \frac{N}{N_\phi} = \frac{1}{m + \frac{\alpha}{p}} \tag{1.15}$$

Adding an electron to the system results in the following changes

$$N_\phi = mN + \alpha(N_{qp} - \alpha m) \tag{1.16}$$

$$N + 1 = p(N_{qp} - \alpha m) + \alpha(mp + \alpha) \tag{1.17}$$

The equation 1.17 amounts to ensuring that the number of quasiparticle states is unaltered by the addition of one electron. The addition of one electron thus produces $(mp + \alpha)$ quasiparticles or quasiholes. These therefore have a charge of $\pm\frac{1}{(mp+\alpha)}$, the reciprocal of the filling fraction.

The hierarchy construction can be iterated so that the quasiparticles and quasiholes at one level in the hierarchy, form the particles and holes of the next level. The general filling fraction for a Laughlin like ground state may be expressed as a

continued fraction

$$\nu = \cfrac{1}{m + \cfrac{\alpha_1}{p_1 + \cfrac{\alpha_2}{p_2 + \cfrac{\alpha_3}{\ddots}}}} \tag{1.18}$$

The natural order in which states appear in the hierarchy seems to agree quite well (at least for the lowest Landau level case) with the apparent 'strength' of the experimental observed plateaux in the Hall resistance. Results from numerical studies for finite size systems (Haldane 1987, d'Ambrumenil and Reynolds 1988, Reynolds and d'Ambrumenil 1988) support the view that the $\nu = \frac{2}{3}, \frac{2}{5}$ and $\frac{3}{5}$ states, the daughter states of the $\nu = \frac{1}{3}$ state are indeed Laughlin like. In Fig.5 I show the hierarchy of FQH states and indicate some experimental observed FQH states.

Zhang 1986 claimed that the effect of impurities rescales and eventually terminates the hierarchy, so that only a finite number of FQH states are quantised in real samples. This is essentially because the energy of a quasiparticle or quasihole scales as

$$\frac{e^{*2}}{\epsilon l_0} = \frac{(e/q)^2}{\epsilon}\left(\frac{e/qB}{\hbar}\right)^{\frac{1}{2}} \sim q^{-\frac{1}{2}} \tag{1.19}$$

where q is given by $\nu = \frac{p}{q}$. So that localisation which precludes the correlation between particles necessary in forming a Laughlin-like ground state occurs more readily in 'higher level fluids'.

Moreover the hierarchy will be terminated by the critical quasiparticle density

below which the gas of quasiparticles at each level of the hierarchy would be expected to form a Wigner crystal so excluding the Laughlin-like fluid from being the ground state.

In an attempt to improve upon the phenomenological footing of the hierarchical picture, Morf et al 1986 suggested a family of microscopic trial wavefunctions, which work explicitly with the electron coordinates, but which appear to incooperate the idea of a hierarchy. Extensive Monte-Carlo calculation of these wavefunctions added further support to the picture of a hierarchy of Laughlin-type fluids.

## 1.8 Cooperative Ring Exchange Theory

An alternative to the Laughlin theory of the FQHE has been proposed by Kivelson et al 1986 (KKAS). They sought to explain the FQHE not in terms of a variational wavefunction but as a lowering of the ground state energy due to cyclic exchange of electrons round large rings. The contributions from these ring exchanges add coherently when the ratio of the number of particles to the number of flux quantum is rational with odd dominator and then preferentially stabilise the system at the corresponding filling fraction.

The original KKAS theory seemed to require that the ground state would have to have a long range solid like order. However Baskaran 1986, later argued that cooperative ring exchange phenomena and the consequent fractionalisation can occur in any incompressible state, including the Laughlin state.

The hierarchy picture and the fractional charge of excitations are found to emerge naturally from the cooperative ring exchange theory. I will return to this description later (chapter 4).

## 1.9 The Central Theme Of This Thesis

The central theme of this thesis is the existence and stability of FQH states other than in the lowest Landau level. The motivation for this study came from the observation of an apparent absence of a well defined FQH state at a filling fraction $\nu = \frac{1}{3}$ in the first (n = 1) Landau level (e.g. see Willett et al 1988, Fig.6). This contrasts with the situation in the lowest (n = 0) Landau level where the FQH state at $\nu_0 = \frac{1}{3}$ is found to be the most robust. This result suggested that the Laughlin theory of FQHE does not generalise directly to higher Landau levels. In addition the observation of a well defined FQH state at $\nu_1 = \frac{2}{3}$ in the first Landau level, the 'daughter' of the $\nu_1 = \frac{1}{3}$ state seemed to cast doubt on the hierarchical picture. This is because taken literally, the hierarchical picture seems to imply that no 'daughter' state can exist unless its 'parent' state also exists.

In chapter 2 I will present the generalisations (Haldane 1987, MacDonald and Girvin 1986) of the Laughlin state to the $n^{th}$ Landau level. I show that for densities $\nu_n^{-1} \leq 1 + 2n$ of the $n^{th}$ Landau level it is not possible to construct a 'physically realisable' interaction for which the Laughlin state would be the exact non-degenerate ground state. The justification for expecting a Laughlin-like ground state at these

densities is therefore weak.

I suggest that a hierachical picture in which a new ground state results from the condensation of a quasiparticle gas from a preceeding level should not be taken too literally. I argue contrary to the hierarchical picture (Chapter 2) that the regime $3 > \nu_1^{-1} > 2$ in the first Landau level is approximately analogous to the regime $2 > \nu_0^{-1} > 1$ in the lowest Landau level, irrespective of whether the ground state is Laughlin like or not at $\nu_1 = \frac{1}{3}$. This claim is supported by numerical studies (Chapter 3) of the collective excitations of electrons in a partially filled first Landau level. I find that at $\nu_1 = \frac{2}{3}$ there is a well defined excitation branch with a roton minimum characteristic of FQH states, despite the non-existence of a Laughlin-like parent fluid at $\nu_1 = \frac{1}{3}$.

I argue (Chapter 4) that a generalised ring exchange theory of the FQHE predicts the occurence of two phase boundaries. This first is between a solid and an incompressible fluid (Wigner crystal and Laughlin fluid) at small fillings $\nu$ of a Landau level. The second corresponds to the boundary between an incompressible and compressible fluid. This second transition occurs only in higher ($n > 0$) Landau levels. The destabilisation of Laughlin-like states in higher Landau levels is shown to emerge naturally in the ring exchange picture. The picture mirrors closely two-stage two-dimensional melting (Kosterlitz and Thouless 1973, Halperin and Nelson 1978, Young 1979).

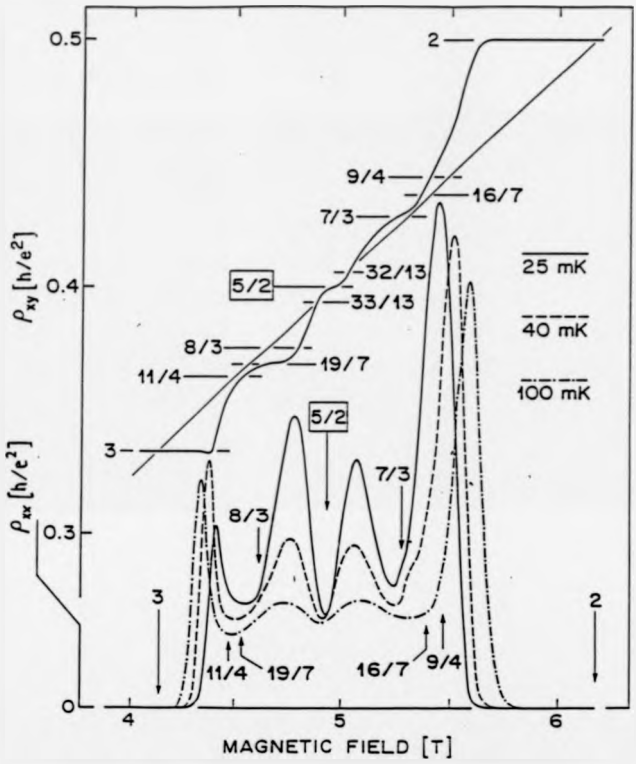


**Figure 6.** Diagonal resistivity $\rho_{xx}$ and Hall resistance $\rho_{xy}$ observed for the first ($n = 1$) Landau level. This figure is an enlargement of section (a) in figure 1.

## 1.10 Summary

The most generally accepted theory of the FQHE is due to Laughlin 1983. In the Laughlin picture the ground state of a system of two-dimensional electrons in a perpendicular magnetic field at densities $\nu^{-1}$ with $\nu^{-1}$ an odd integer, is that of an incompressible fluid. A hierarchical picture (Haldane 1983, Halperin 1984) has been proposed to account for FQH states with $\nu^{-1}$ non integer. An alternative understanding of the FQHE has been suggested by KKAS 1986.

In the remainder of this thesis I discuss the existence and stability of FQH states, the hierarchy picture and the cooperative ring exchange theory of the FQHE for other than the lowest Landau level.

# Chapter 2

# The Nature Of The Laughlin State In Higher Landau Levels

## 2.1 Introduction

In this chapter the existence and stability of spin polarized Laughlin fluids other than in the lowest ($n = 0$) Landau level is investigated. The motivation for this work came from the apparent absence of a well defined FQH state at $\nu_1 = \frac{1}{3}$ in the first ($n = 1$) Landau level, (e.g. see Willett et al 1988, Fig.6), which seems to suggest that the ground state at this density may not be Laughlin-like.

I discuss the sort of interactions which favour a Laughlin-like ground state. I show that for densities $\nu^{-1} \leq 2n + 1$ of the $n^{th}$ Landau level, generalisations of the Laughlin state to higher Landau levels are the exact ground state for a 'physically

unrealisable' interaction between electrons, suggesting that at these densities the ground state is not well described by a Laughlin-like state. This view is supported by results from numerical studies for small systems of electrons (d'Ambrumenil and Reynolds 1988, Haldane 1987).

I argue that the regime $2n + 3 > \nu_n^{-1} > 2n + 1$ in the $n^{th}$ Landau level is approximately analogous to the regime $3 > \nu_0^{-1} > 1$ in the lowest Landau level and suggest that the hierarchical picture should not be taken too literally. Numerical calculations (presented in Chapter 3) for small systems of electrons in the first Landau level support this view.

The remainder of this chapter is organised as follows. In the next section, I demonstrate that the Laughlin state is the exact non-degenerate ground state for a kind of 'hard-core' interaction. In section 2.3I discuss the generalisations of the Laughlin state to higher ($n > 0$) Landau levels. I show that there is an analogy between these states and Laughlin states in the lowest Landau level. Then in section 2.4 I show that for densities $\nu^{-1} \leq 2n + 1$ it is not possible to construct a 'physical interaction' for which the Laughlin states are the exact non-degenerate ground states. I discuss the nature of the hierarchy in higher ($n > 0$) Landau levels in section 2.5. A summary is presented in section 2.6.

## 2.2 The Exactness of the Laughlin State

The Laughlin states $\Phi_L(z_1, \ldots, z_N)$ may be written as

$$\Phi_L(z_1, \ldots, z_N) = \prod_{i<j}^{N} (z_i - z_j)^m \mid 0 > \qquad m = 1, 3, 5, \ldots, \tag{2.1}$$

where $z_i = x_i + i(\frac{\ell^2}{\hbar})p_i$ is the coordinate of the $i^{th}$ particle and m is the relative angular momentum quantum number for pairs of particles i and j. $\mid 0 >$ is the product state over the N single-particle orbitals in the lowest Landau level with angular momentum, set to zero. In the symmetric gauge $\mathbf{A}(r_j) = (\mathbf{B} \times r_j/2)$,

$$\mid 0 > = \prod_{j}^{N} \exp\left(- \mid z_j \mid^2 /4\right) \tag{2.2}$$

The Laughlin states at $\nu = \frac{1}{m}$ have been shown (Halperin 1983, Haldane 1983, Trugman and Kivelson 1985, Pokrovsky and Talapov 1985) to be the exact non-degenerate ground state of a kind of 'hard-core' interaction. This is perhaps most readily seen for rotationally invariant systems, for which the interparticle interaction can be written as

$$v(i,j) = \sum_{m} V_m P_m(i,j). \tag{2.3}$$

Here $P_m(i,j)$ projects out of the many body wavefunction, those components with relative angular momentum m, for particles i and j.

The essential feature of the Laughlin wavefunction $\Phi_L$ is that it contains no components with relative angular momentum less than m. At $\nu = \frac{1}{m}$ it is the only wavefunction with this property and so is the exact non-degenerate ground state for

an interaction

$$V_{m'} = 1 \qquad m' < m$$

$$V_{m'} = 0 \qquad m' \geq m \tag{2.4}$$

The reason why such an interaction may be called 'hard-core' can be seen by considering the pair correlation function $g(r)$. The Laughlin state is the only wavefunction in which all terms in the pair correlation function rising faster than $r^{2m}$ for small $r$ are suppressed, see fig. 7. That is, in a system described by $\Phi_L$ the probability for any two particles (i and j) approaching one another vanishes as

$$|\Phi(r_i, r_j)|^2 \sim |r_i - r_j|^{2m} \tag{2.5}$$

Haldane 1987, imagined a Gedankenexperiment in which the range of the interaction was adiabatically changed from this 'hard-core' interaction to the interaction of the physical system. Provided that no cross-over occurred between states of different symmetry, the physical system would be qualitatively the same as the system described by the Laughlin state.

Haldane investigated the adiabatic variation of the interaction by direct numerical diagonalisation of the Hamiltonian for small systems. He defined an interaction

$$V(\lambda) = V^{H.C.} + \lambda \left( V^{COULOMB} - V^{H.C.} \right) \tag{2.6}$$

and adiabatically varied $\lambda$ from $\lambda = 0$ where the interaction $V(\lambda) = V^{H.C.}$ to beyond $\lambda = 1$. At $\lambda = 1$, $V(\lambda) = V^{COULOMB}$. Here $V^{COULOMB}$ describes the Coulomb interaction and $V^{H.C.}$ the 'hard-core' interaction.

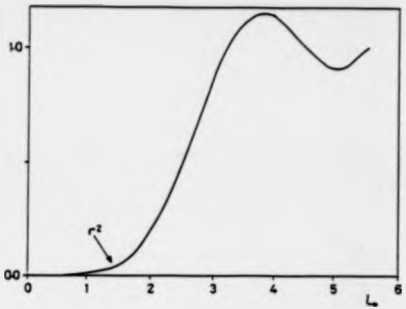


**Figure 7.** The Laughlin ground state pair correlation function for the $\nu_0 = \frac{1}{3}$ six particle system on the sphere. The short range behaviour $r_{ij} \to 0$ is indicated.

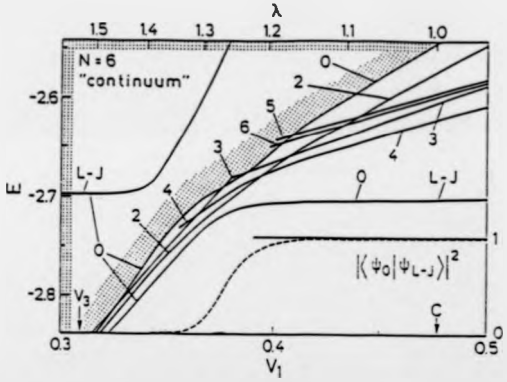


**Figure 8.** Low lying states at $\nu_0 = \frac{1}{3}$ for the six particle system on the sphere, as the component $V_1$ of the hardcore interaction is varied. The other $V_m$ take on their Coulomb values. The Coulomb value of $V_1$ (marked as C) and $V_3$ is shown. The values of angular momentum L are indicated. Also shown is the projection of the Laughlin state onto the exact ground state.

In fig.8 I reproduce the results of Haldane, for the ground state and excited states of the six particle system at a filling $\nu_0 = \frac{1}{3}$. Fig. 8 shows that at least for the six particle system there is no cross-over between states with different quantum numbers for the range of interactions likely in the physical system. For $\lambda \leq 1.25$ the Laughlin state has almost 100% projection onto the true ground state, and is separated by an excitation gap from the first excited state. This range includes the ground state for the pure Coulomb interaction, which is therefore well described by the Laughlin state.

Although rotational invariance was invoked in constructing the 'hard-core' interaction, it is not fundamental to the FQHE. The FQHE survives impurities, which break rotational invariance. Rotational invariance is removed if the effective mass tensor is anisotropic. It has been shown (Haldane 1987) that for sufficently weak effective mass anisotropy the character of FQH states of the pure system are unchanged by breaking rotational invariance.

## 2.3 Laughlin States in Higher Landau Levels

An obvious way (Haldane 1987, MacDonald and Girvin 1986) to generalise the Laughlin states to arbitrary Landau level index n, is to replace the particle coordinates $z_j$ in $\Phi_L$ by the raising operators $Z_j^+$ for angular momentum $L_z^j$.

These generalised Laughlin states $\Phi_L^n(r_1, \ldots, r_N)$ take the form

$$\Phi_L^n(r_1, \ldots, r_N) = \prod_{i<j}^{N} \left(Z_i^+ - Z_j^+\right)^m |0>_n \qquad m = 1, 3, 5, \ldots, \qquad (2.7)$$

where $| 0 >_n$ is the product state over the N single-particle orbitals in the $n^{th}$ Landau level with angular momentum, $L_z^i$, equal to zero.

One may ask about the kind of interactions which would give rise to a Laughlin-like ground state in higher Landau levels. I will show in the next section that for fillings $\nu_n$ with $\nu_n^{-1} \leq 2n + 1$ this interaction cannot be defined in real space. In what follows I discuss why one might expect 'something to go wrong' with the construction of Laughlin-like states in higher Landau levels.

In higher Landau levels ($n \neq 0$) $\Phi_L^n$ has components

$$\Phi_L^n \sim L_M^{2l+m}\left(|r_{ij}|^2/4\right)(z_i - z_j)^{m-2l}\exp\left(-|r_{ij}|^2/8\right) \qquad (2.8)$$

with $l \leq n$ for $|r_{ij}| \to 0$. Here the $L_j^i$ are the Laguerre polynomials and $z_i = x_i - iy_i$.

The component with $l = n$ rises most slowly for small $r_{ij}$. One may show that for a Laughlin-like ground state, in the limit $r_{ij} \to 0$

$$|\Phi_L^n|^2 \sim |r_{ij}|^2 + 0\left(|r_{ij}|^6\right) \quad \text{for} \quad m \leq 1 + 2n \qquad (2.9)$$

$$|\Phi_L^n|^2 \sim |r_{ij}|^{2(m-2n)} + 0\left(|r_{ij}|^{2(m-2n+2)}\right) \quad \text{for} \quad m > 1 + 2n \qquad (2.10)$$

so that for all $\nu_n^{-1} < 1 + 2n$ the pair correlation function, $g(r)$, always rises as $r^2$. In section 2.2 I showed that for the lowest Landau level, two dimensional electrons in a perpendicular magnetic field can condense into a Laughlin ground state whenever the variation with $\nu_n^{-1}$ of the pair correlation function has a discontinuity of slope. If the correspondence carries over to higher Landau levels then the justification for expecting a Laughlin-like ground state at densities with $\nu_n^{-1} < 3 + 2n$ is weak.

For $m = 1 + 2n + 2l$, $l = 1, 2, ...$, $\Phi_l^n$ is the only state that has no component in $g(r)$ rising faster than $r^{2+2l}$. At $\nu_n^{-1} = 1 + 2n + 2l$, $\Phi_l^n$ is therefore the exact non-degenerate ground state for the same 'hard-core' interaction for which $\Phi_l^0$ is the exact ground state at $\nu_0^{-1} = 1 + 2l$.

Just as was suggested by Haldane 1987 for the lowest Landau level, one can imagine adiabatically changing the range of the interaction from the 'hard-core' like interaction for which $\Phi_l^n$ is the exact ground state at $\nu_n^{-1} = 1 + 2n + 2l$ to the interaction of the physical system. Provided no cross over occured between states of different symmetry the properties of the physical system would be qualitatively similar to those for which $\Phi_l$ is exact. That is, the ground state should be that of an incompressible fluid separated by a gap from fractionally charged excitations. At $\nu_n^{-1} = 1 + 2n + 2l$ one would therefore expect the ground state to be Laughlin-like.

## 2.4 The Hamiltonian and Reconstruction of the Interaction

Given the Laughlin state or one of its generalisations I ask, what interaction between electrons would give rise to such a state. I will show that for densities $\nu^{-1}$, with $\nu^{-1} \leq 2n + 1$ the generalisations of the Laughlin state to the $n^{th}$ Landau level (Haldane 1987, MacDonald and Girvin 1986) are the exact ground state for an interaction between electrons which is physically unrealisable. The justification for expecting a Laughlin-like ground at these densities is therefore weak.

As before I assume that in high magnetic fields and at low temperatures, the cyclotron orbit kinetic energy of electrons exceeds characteristic thermal and potential energies. Then processes which do not conserve the Landau level index can therefore be neglected, and the dynamics of the electrons can be described in terms of a single Landau level index. The Hamiltonian for a system of two dimensional electrons confined to the $n^{th}$ Landau level may be written (Haldane 1987) as

$$\begin{aligned} H &= H_0 + \sum_{i<j} H_{ij}, \\ H_0 &= \left(n + \frac{1}{2}\right)\hbar\omega_c, \\ H_{ij} &= \frac{1}{2\pi}\int d^2Q V(Q) \exp i\mathbf{Q}.(\mathbf{r}_i - \mathbf{r}_j). \end{aligned} \tag{2.11}$$

Here $\omega_c = (eB/m)$ is the cyclotron frequency. The matrix elements $H_{ij}$ are taken only between states that have both electrons i and j in the $n^{th}$ Landau level.

The dynamics of electrons in the FQHE are perhaps most readily understood by eliminating their cyclotron motion, and studying the quantisation of the 'guiding centres'. The guiding centre coordinates $\mathbf{R}_i$ for an electron in the xy plane at position $\mathbf{r}_i$ with momentum $\pi_i$ are

$$\mathbf{R}_i = \mathbf{r}_i - \mathbf{a} \times \pi_i \tag{2.12}$$

which classically are just the centres the cyclotron motion.

The Hamiltonian $H_{ij}$ describing the residual interaction may be recast in terms of these guiding centre coordinates by writing

$$H_{ij} = \frac{1}{2\pi}\int d^2Q V(\mathbf{Q}) \mid k_1, k_2 > < k_1, k_2 \mid e^{i\mathbf{Q}.\mathbf{R}_{ij}} \mid k_3, k_4 > \times \tag{2.13}$$

$$< k_3, k_4 \mid e^{iQ.R_{ij}} \mid k_5, k_6 >< k_5, k_6 \mid .$$

Here the $\mid k_i, k_j >$ are normalised two particle states in the $n^{th}$ Landau level, formed from single particle states $\mid n, k >$ which in the Landau gauge are (Appendix A)

$$\mid n, k >= e^{-ik.z} e^{-(y-k)^2/2} H_n(y-k). \tag{2.14}$$

One finds (after a little algebra) that

$$< n, k \mid e^{-iQ.z} \mid n, k' >= \delta_{k,k'} L_n\left(Q^2/2\right) \exp\left(\frac{1}{2}Q^2\right) \tag{2.15}$$

so that the interaction may be written as

$$H_{ij} = \frac{1}{2\pi} \int d^2Q V(\mathbf{Q}) \exp\left(-\frac{1}{2}Q^2\right) \left[L_n\left(Q^2/2\right)\right]^2 \exp\left[i\mathbf{Q}.\left(\mathbf{R}_i - \mathbf{R}_j\right)\right]. \tag{2.16}$$

In a rotationally invariant system the interaction Hamiltonian, $H_{ij}$, may be written as

$$H_{ij} = \sum_{m=0}^{\infty} V_m^n P_m(i,j) \tag{2.17}$$

where $V_m^n$ is the energy of a pair of electrons in the $n^{th}$ Landau level with relative angular momentum m and where $P_m(i,j)$ projects out from some wavefunction components with relative angular momentum $M_{ij} = m$. (For a discussion of the definition of angular momentum in a magnetic field see appendix B.)

In terms of the guiding centre coordinates $\mathbf{R}_i$, $M_{i,j}$ and $P_m(i,j)$ are found to take the form

$$M_{i,j} = \mid \mathbf{R}_i - \mathbf{R}_j \mid^2 /4 - \frac{1}{2} \tag{2.18}$$

$$P_m(i,j) = \frac{1}{\pi} \int d^2Q L_m\left(Q^2\right) \exp\left(-Q^2/2\right) \exp\left[i\mathbf{Q}.\left(\mathbf{R}_i - \mathbf{R}_j\right)\right] \tag{2.19}$$

Once the $V_m^n$ are known the Hamiltonian for electrons confined to the $n^{th}$ Landau level is completely defined.

In sections 2.3 and 2.4 I showed that one can find an interaction defined in terms of the $V_m^n$ for which $\Phi_L^n$ is the exact non-degenerate ground state. Given such a set of $V_m^n$ I ask what interaction between electrons $v(|\,r_{ij}\,|)$ might give rise to this set. The $V_m^n$ alone do not define the original interaction $v(|\,r_{ij}\,|)$ exactly. For this terms mixing Landau levels, $V_m^{nn'}$ and $V_m^{n'}$ for Landau levels $n \neq n'$ are necessary. I 'reconstruct' $v(|\,r_{ij}\,|)$ from the $V_m^n$ as follows and assume that the $V_m^{nn'}$ and $V_m^{n'}$ take on values consistent with the interaction $v(|\,r_{ij}\,|)$ obtained. Inserting eqn.2.19 into eqn.2.17 and comparing with eqn.2.16 I make the identification

$$V(Q) = \sum_m V_m L_m\left(Q^2\right) / \left[L_n\left(Q^2/2\right)\right]^2 \tag{2.20}$$

Now it is seen that truncated interaction, $V_m^1 = \delta_{1,m}$ for which $\Phi_L^1$ would be the exact non-degenerate ground state at filling fraction $\nu_1 = \frac{1}{3}$, corresponds to an interaction $V(Q) = (1-Q^2)/(1-Q^2/2)^2$. This corresponds to an interaction $v(|\,r_{ij}\,|)$ that is not defined.

In fact any truncated interaction of the form

$V_m^n = a_m \quad m < b_n$

$V_m^n = 0 \quad m \geq b_n$

will have poles in $V(Q)$ for all $b_n < 3+2n$. For $b \geq 3+2n$ it is always possible to construct truncated interactions corresponding to 'hard-core' interactions for which $\Phi_L$ is the exact non-degenerate ground state.

I assert that the absence of a well defined FQH state at $\nu_1 = \frac{1}{3}$ (see e.g. Willett

et al, Fig.6) is a consequence of the ground state not being Laughlin-like at this filling. Numerical studies of systems with up to nine electrons confined to the surface of a sphere (d'Ambrumenil and Reynolds 1988) and of collective excitations (Reynolds and d'Ambrumenil 1988 and Chapter 3) support this view. We found that at $\nu_1 = \frac{1}{3}$ the ground state is not well described by a Laughlin-like ground state and is close to the boundary between being compressible and incompressible. Similar results were obtained by Haldane 1987, using periodic boundary conditions and numerically diagonalising the Hamiltonian for six electrons. The gap in the excitation spectrum is predicted to be small or non-existent.

## 2.5 The Hierarchy in Higher Landau Levels

In this section I consider the validity of the hierarchy picture of the FQHE and ask whether or not the absence of a 'parent' state necessarily implies the absence of a 'daughter' state.

I consider this question because recently observations of a well defined FQH state at $\nu_1 = \frac{2}{7}$ in the first Landau level, (which in the hierarchy scheme would be the daughter state of the $\nu_1 = \frac{1}{3}$ FQH state) have been reported (Clarke et al 1988 and Willett et al 1988, Fig.6).

Numerical studies for $\nu_1 = \frac{2}{7}$ (d'Ambrumenil and Reynolds 1988, Reynolds and d'Ambrumenil 1988, see Chapter 3) suggest that the ground state is indeed Laughlin-like, with a gap and roton minima in the collective excitation spectrum characteristic

of FQH states, despite the apparent 'marginal stability' of the parent $\nu_1 = \frac{1}{3}$ ground state. This result seems to imply that a heirarchical picture should not be taken too literally.

An alternative view (d'Ambrumenil and Reynolds 1983) to the hierarchy may be argued as follows. A system of two dimensional electrons in a perpendicular magnetic condenses into the Laughlin state whenever the variation with $\nu_0^{-1}$ of the component in the pair-correlation function rising as $r_{ij}^{2(2l+1)}$ has a discontinuity of slope (see section 2.3). For $\nu_0^{-1} > 1 + 2n + 2l$ this component may be zero while for $\nu_0^{-1} < 1 + 2n + 2l$ it is greater than zero. At intermediate values of $\nu_0^{-1}$ the system may condense into some fluid-like state where there is again the possibility of a discontinuity in the slope of the component rising as $r_{ij}^{2(2l-1)}$ even though the value of the component remains non-zero.

In section 2.4 I argued the ground state of the physical system was expected to be Laughlin-like at filling fractions with $3 + 2n > \nu_0^{-1} > 1 + 2n$, in approximate analogy to ground states with filling fractions $3 > \nu_0^{-1} > 1$. If plateaux are related to discontinuities in the slope of the variation of the pair correlation function with filling fraction then one would expect this analogy to hold independent of whether the $\nu_0^{-1} = 1 + 2n$ state is Laughlin-like or not.

## 2.6 Summary

In this chapter I argued that for densities with $1 + 2l + 2n < \nu_0^{-1} < 3 + 2l + 2n$ the generalisations of the Laughlin state are approximately analogous to the Laughlin states in the lowest Landau level with $1 + 2l < \nu_0^{-1} < 3 + 2l$. This claim is supported by the results of numerical studies (presented in chapter 3) for small systems of electrons. Provided that they are not obscured by the normal quantum Hall effect or impurity effects, I would expect to see plateaux in the Hall resistance at those densities with $5 < \nu_1^{-1} < 3$ analogous to the densities in the lowest Landau level with $3 < \nu_0^{-1} < 1$, independent of whether the ground state is Laughlin-like at $\nu_1^{-1} = 3$ or not. I have also shown that it is not possible to construct a 'physical interaction' for which the Laughlin-like state $\nu_1 = \frac{1}{3}$ is the exact non-degenerate ground state. These claims are supported by the results of numerical calculations presented in chapter 3.

# Chapter 3

# Collective Excitations In The Fractional Quantum Hall Effect

## 3.1 Introduction

In this chapter, I study the dispersion of collective excitations for Laughlin-like states using Feynman-Bijl ansatz wavefunctions.

The low-lying excitations of the Laughlin state can be thought of as density waves, Girvin MacDonald and Platzman (GMP) 1986. Making the approximation that only one 'band' of such collective excitations coupled to the ground state they, were able to predict the dispersion of this band variationally.

GMP proposed using an adapted Feynman-Bijl ansatz to describe the collective excitations at wavevector k in the lowest Landau level. They considered the trial

wavefunction formed by operating on the homogeneous ground state, (which is assumed known), with the density operator projected onto the lowest Landau level. They were then able to relate the dispersion, $\Delta(k)$, of the collective excitation to the pair correlation function, $g(r)$, for particles in the ground state.

The method of GMP gives a numerical estimate for the dispersion, $\Delta(k)$, which is in good agreement with exact numerical results for small systems (Haldane and Rezayi, 1985). It has also been generalised to higher Landau levels (MacDonald and Platzman, 1986), where it has been found to agree with exact numerical results for small systems when the ground state is known to be homogeneous and Laughlin-like (d'Ambrumenil and Reynolds 1988).

This elegant and simple method has so far been restricted to systems of electrons at filling fractions of the lowest Landau level, $\nu^{-1} = m$, where m is an odd integer. This is because it is only at these filling fractions that the Laughlin wavefunction exists so that a good approximation to the pair correlation function, $g(r)$, can be found using Monte-Carlo methods.

It would be nice to generalise the method to filling fractions at which a system of electrons is expected to have a Laughlin-like incompressible ground state and for which generalisations of the Laughlin wavefunction have been suggested (Morf et al, 1986).

Here I consider the possibility of taking the pair correlation function, which forms the basis for the variational estimate for collective excitations, from the results

obtained from diagonalisations of the Hamiltonian for small systems. Clearly this will lead to problems for large separations, (small wave numbers), as the finite size effects will be larger than those encountered by GMP. ( The pair correlation function for Laughlin's wavefunction is known for systems with up to 256 particles, whereas finite size calculations obtained from diagonalisation of the Hamiltonian are only possible for systems with up to twelve particles).

I also study the excitations in the first (N=1) Landau level. It is known that for filling fractions $\nu_1^{-1} \geq 5$ in the N=1 Landau level the Laughlin theory generalises from the N=0 case without problems (see Chapter 2). I suggested in chapter 2 that the regime $3 < \nu_1^{-1} < 5$ was approximately equivalent to the regime $1 < \nu_0^{-1} < 3$. I therefore would like to study collective excitations in this regime.

The remainder of this chapter is organised as follows. In section 3.2 I present a derivation of the dispersion in the collective excitation spectrum using the Feynman-Bijl ansatz of Girvin et al. I discuss how the pair correlation function taken from results of diagonalisation of the Hamiltonian for small systems might be used in such calculations. I present results of numerical calculations performed for systems in the lowest (N=0) and first (N=1) Landau levels in section 3.3. A discussion on the nature of the collective spectrum at large wavevectors is given in section 3.4. Section 3.5 is a summary.

## 3.2 Collective Excitation Spectrum

GMP suggested that the low-lying collective modes were density waves. In the spirit of the Feynman-Bijl theory of liquid helium they considered a trial wavefunction,

$$\Phi_k = \bar{\rho}_k^N \mid N > \tag{3.1}$$

for the collective excitation with wavevector k . Here $\bar{\rho}_k^N$ is the density operator projected onto the N'th Landau level and $\mid N >$ is the homogeneous ground state for a system of electrons or holes in the N'th Landau level. GMP showed that the variational estimate for the energy of collective excitations,

$$\Delta(k) = \frac{< \Phi_k \mid H - E_0 \mid \Phi_k >}{< \Phi_k \mid \Phi_k >}, \tag{3.2}$$

with $E_0$ the ground state energy, is in fact the first moment of the spectral function for intra-Landau level excitations.

The projected density operator acts in part like a translation operator (see section 3.4), translating particles through a distance $\mid k \mid l_0^2$, so creating particle-hole pairs. The centre of mass momentum k of this neutral excitation does not couple to the magnetic field and so is conserved independent of gauge. This centre of mass momentum is used to label the excited states.

The assumption that the density wave, $\Phi_k$, is a good trial wavefunction is equivalent to assuming that only one collective mode with wavevector k couples to the ground state, $\mid N >$. The approximation is therefore sometimes known as the single

mode approximation (SMA). The SMA is exact in the limit of vanishingly small k , (see GMP), but starts to break down at finite wavevectors. This is because the density wave is not a sensible excitation for wavelengths smaller than the interparticle spacing.

GMP and later MacDonald and Girvin 1986 showed that $\Delta(k)$ was given by

$$\Delta(k) = \frac{1}{2}\sum_{q} v(q)\left(e^{q^2/2} - e^{ik^2/2}\right)\overline{S}(q)e^{-k^2/2}\left(e^{-ik\times q/2} - e^{-ik\times q/2}\right) + \overline{S}(k+q)\left(e^{ik\times q/2} - e^{ik\times q/2}\right)\left(L_N(q^2/2)\right)^2 / \overline{S}(k) \quad (3.3)$$

where the projected structure factor defined by

$$\overline{S}^N(k) = < N \mid \overline{\rho}_k^N \overline{\rho}_k^N \mid N > \quad (3.4)$$

is essentially just a modified Fourier transform of the pair correlation function $g_N(r)$ for particles in the ground state $\mid N >$. I independently derived this result (unpublished) prior to knowledge of MacDonald and Platzman result.

The evaluation of $\Delta(k)$, the variational estimate of the collective excitation energy, essentially reduces to finding $\overline{S}^N(k)$. GMP took $\overline{S}^N(k)$ to be that for particles described by Laughlin's trial wavefunction. They showed that for $N \neq 0$, $\overline{S}^N(k)$ could be related to an effective pair correlation function $g'(r)$ for particles in a state $\mid 0 >$ , where $\mid 0' >$ is the 'image' of the ground state $\mid N >$ in the lowest Landau level.

They write

$$\bar{\rho}_k^N = L_N\left(k^2/2\right)\bar{\rho}_k, \tag{3.5}$$

$$\bar{\rho}_k = \sum_i e^{ik\cdot R_i} \tag{3.6}$$

with

$$R_i = r_i + z \times \Pi_i \tag{3.7}$$

the cyclotron orbit center operator for the i'th electron. They also define Landau level raising and lowering operators for the i'th particle $A_i^\pm$

$$A_i^\pm = \frac{X_i \pm i Y_i}{\sqrt{2}} \tag{3.8}$$

The operators $A_i^\pm$ commute with the density-like operators $\bar{\rho}_K$.

The 'image' state $|\,0'>$ is then just

$$|\,0'> = \prod_{i=1}^{N} (A_i^-)^N \,|\, N>, \tag{3.9}$$

and the projected structure factor $\bar{S}^N(k)$ can be written as

$$\begin{aligned} \bar{S}^N(k) &= \left[L_N\left(k^2/2\right)\right]^2 < N \,|\, \bar{\rho}_k^+ \bar{\rho}_k \,|\, N >, \\ &= \left[L_N\left(k^2/2\right)\right]^2 < 0' \,|\, \bar{\rho}_k^+ \bar{\rho}_k \,|\, 0' >, \\ &= \left[L_N\left(k^2/2\right)\right]^2 \bar{S}^{0'}(k). \end{aligned} \tag{3.10}$$

Equation 3.10 shows that the projected structure factor in the N'th Landau level is equal to the one the system would have in the lowest Landau level if it were described by the state $|0'>$, but modified by factors $L_N(k^2/2)^2$.

My objective was to study the SMA at filling fractions, $\nu \neq 1/m$, estimating $\overline{S}^N(k)$ from the exact ground state wavefunction $|N>$, calculated for small systems. My procedure involves evaluating the pair correlation function $g_{0'}(r)$ which is the Fourier transform of $S^{0'}(k)$ (equation 10)

$$S^{0'} = 1 + \rho \int d^2R e^{-ik.R}[g_{0'}(R) - 1] \tag{3.11}$$

for the finite system and fitting to the form it must have in the infinite system to satisfy various sum rules (GMP). These sum rules amount to ensuring charge neutrality, perfect screening and incompressibility in the 2DOCP analog of the Laughlin state. The extrapolation to infinite systems essentially occurs at this stage. We then estimate $\overline{S}^{0'}(k)$ for the infinite system from the relation 3.11 and insert this into the equation 3.10 to give $\overline{S}^N(k)$.

My evaluations of the ground state $|N>$ are for systems of particles confined to the surface of a sphere. The advantage of the sphere over other geometries is that it ensures a homogeneous electron distribution and leads to no boundary effects. For systems of particles on a sphere, however, it is not possible to define explicit raising and lowering operators $A^{\pm}_i$. (This is related to the fact that for a fixed magnetic charge at the centre of the sphere, $2S$, there are $2(S+N)+1$ single-particle states

in the $N^{th}$ Landau level, so that the number of cyclotron orbits depends on the Landau level index, N.)

Because there are no corresponding operators to the $A_i^\dagger$ for systems on a sphere it is not possible to define the 'image' state $|\ 0'>$ as in equation 3.8. Instead I define $|\ 0'>$ as follows. I say $|\ 0'>$ has the same expansion in terms of Slater determinants of single-particle states labeled by angular momentum $|\ l_z>$ as the state $|\ N>$. The ground state $|\ N>$ for the system with magnetic charge at the centre of the sphere (Haldane, 1983), 2S may be expanded in the basis of Slater determinants of single-particle states:

$$l_z = -S-N, -S-N+1, \ldots, S+N-1, S+N \qquad (3.12)$$

The 'image' state $|\ 0'>$ is then said to have the same expansion in the basis

$$l_z = -S^0, -S^0+1, \ldots, S^0-1, S^0, \qquad (3.13)$$

where $S^0 = S+N$

This definition ensures that $|\ 0'>$ and $|\ N>$ have the same angular momentum quantum numbers, (L=0). It also ensures that the Laughlin trial wavefunction for systems in different Landau levels (d'Ambrumenil and Reynolds, 1988) all have the same 'image' state.

I first establish the validity of my procedure by studying the variational estimate of $\Delta(k)$, for cases already studied by MacDonald and Girvin 1986. I look at the

case $\nu_0^{-1} = 3$ and study the estimates of $\Delta(k)$ and their dependence on system size. I then proceed to study other fractional filling fractions in both the lowest and the N=1 Landau level. The results are presented in the next section.

## 3.3 Results Of Numerical Studies

**Validation of the Method**

In Fig.9 I show the pair correlation function $g(r)$ for the six, seven and eight particle ground states at $\nu_1 = \frac{1}{3}$ for the coulomb interaction. I found that the pair correlation function vanishes as $r_{ij}^2$ as the particles i and j come together ($r_{ij} \to 0$). This behaviour is characteristic of a Laughlin-like ground state for which, for all $\nu_0^{-1} \leq 1 + 2n$ the pair correlation function always rises as $r_{ij}^2$ as $r_{ij} \to 0$ (section 2.2). This suggests that despite the absence of a well defined roton minimum in the collective exicitation (see fig.11) characteristic of FQH states the ground state at $\nu_1 = \frac{1}{3}$ is close to being Laughlin-like. This adds support to the claim made in chapter 2 that the Laughlin state at $\nu_1 = \frac{1}{3}$ is only marginally stable.

In the systems discussed above, the oscillatory nature of the pair correlation function is apparent, with the eight particle system being large enough to show a secondary maximum. However the large $r_{ij}$ dependance of the pair correlation function is somewhat size dependent, a consequence of curvature. The pair correlation functions were found to be more sensitive to finite size effects than their Fourier transforms, the structures factors, Fig.10. These structure factors were obtained

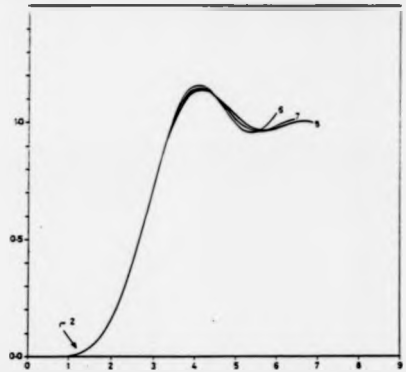


**Figure 9.** The exact (Coulomb interaction) ground state pair correlation functions for the $\nu_1 = \frac{1}{3}$ six, seven and eight particle systems on the sphere. The size dependence is a consequence of surface curvature in the spherical geometry. The short range behaviour $r_{ij} \to 0$ is indicated.

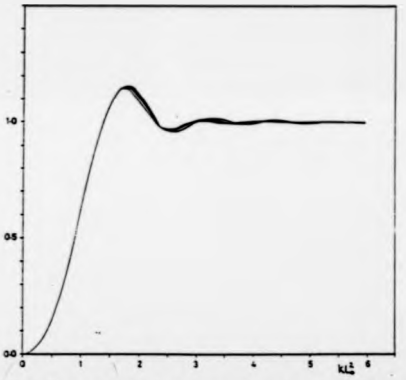


**Figure 10.** The exact (Coulomb interaction) ground state structure factors for the $\nu_1 = \frac{1}{3}$ six, seven and eight particle systems on the sphere. The size dependence is a consequence of surface curvature in the spherical geometry.

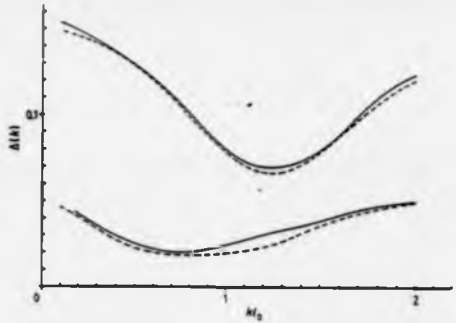


**Figure 11.** The variational estimate of the collective excitation dispersion, $\Delta(k)$, at $\nu_0 = \frac{1}{3}$ and $\nu_1 = \frac{1}{3}$ using the Laughlin eight particle ground state. Energies are in units of $e^2/\epsilon_0 l_0$. The results of MacDonald and Platzman 1986 (—) are shown for comparison.

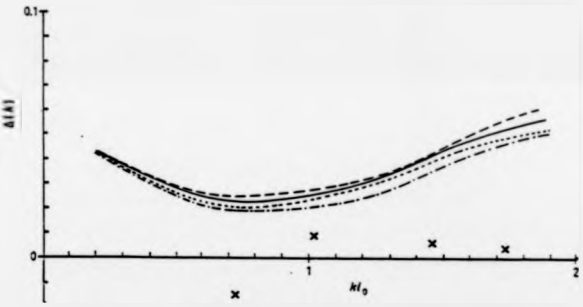


**Figure 12.** The variational estimate of the collective excitation dispersion, $\Delta(k)$, at $\nu_1 = \frac{1}{3}$ for five(- -), six(—), seven(· ·) and eight(· — ·) particle systems, using the exact ground states. Energies are in units of $e^2/\epsilon_0 l_0$. The excitation energies for the five particle system (×) are shown for comparison with the SMA. These energies are defined as the total energy differences between the lowest energy states for the various L and the exact ground state.

using the prescription described in section 3.2.

Figure 11 shows the variational estimates for $\Delta(k)$ obtained using the Laughlin ground state wavefunction for eight particles at $\nu_0 = \frac{1}{3}$ and $\nu_1 = \frac{1}{5}$. All the energies are given in terms of $e^2/\epsilon l_0$, where $l_0$ is the magnetic length of the infinite system. Also shown are the variational estimates obtained by MacDonald and Platzman (1986) using the Laughlin trial wavefunction for 256 particles. I take the high level of correspondence between the two sets of curves to be confirmation of the validity of our numerical procedure.

I found numerical difficulties in determining $\Delta(k)$ for $k < 0.3$. This is expected as these values of k correspond to distances larger than the size of the system, for which our estimate of g(r) should not be trusted. This is of course a problem at all filling fractions.

Figure 12 shows $\Delta(k)$ predicted using the pair correlation function, $g_0(r)$, taken from the homogeneous state with the lowest energy for systems of particles at $\nu_1 = \frac{1}{5}$. Note that this homogeneous state is not the ground state for the system with five particles on the sphere taking the separation between particles as the chord distance (d'Ambrumenil and Reynolds 1988). This is apparent in the figure where the exact excitation energies of the lowest energy states with finite total angular momentum are also shown. The L=2 state has a negative excitation energy with respect to the homogeneous (L=0) state (d'Ambrumenil and Morf 1988). All energies are plotted as a function of total angular momentum rescaled to give the effective wavenumber,

$K_{eff}$.

$$K_{eff} = L/R \tag{3.14}$$

where the radius of the sphere, $R$, is measured in units of $l_0$.

Intriguingly the instability of the homogeneous ground state in the five particle system does not show up in the dispersion, $\Delta(k)$, predicted using the SMA. Although this may be because it is lost in the fitting of $g_0(r)$ to a form for $g_0(r)$ valid for the infinite system, it could also be because the alternative (inhomogeneous) ground state is not coupled via a density wave excitation to the homogeneous Laughlin-like state. If this were true it would seem to imply that the transition from Laughlin-like fluid ground state to inhomogeneous ground state would not be continuous as some control parameter such as the form of the interparticle interaction were varied.

**Prediction of the dispersion at other filling fractions**

In this section I present the results of applying the method of Girvin and Platzman to filling fractions other than $\nu^{-1} = m$. Again I found excellent scaling with system size and good agreement between the predicted dispersion for the infinite system and the exact results in the finite system, except possibly at $\nu_0 = \frac{2}{5}$.

**Results at $\nu_0 = \frac{2}{5}$**

The excitation spectrum for this state has been investigated by Su and Wu,

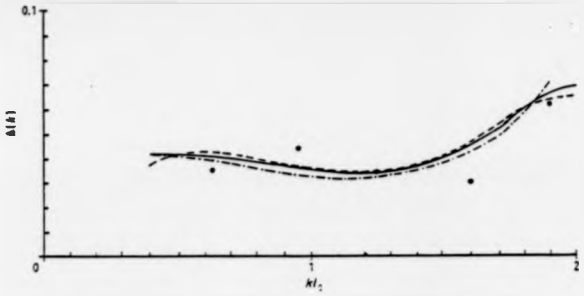


**Figure 13.** The variational estimate of the collective excitation dispersion, $\Delta(k)$, at $\nu_0 = \frac{2}{5}$ for six(- -), eight(—) and ten(—·) particle systems. Energies are in units of $e^2/\epsilon_0 l_0$. The excitation energies for the ten particle system (•) are shown for comparison with the SMA. These energies are defined as the total energy differnce between the lowest energy state for various L and the ground state.

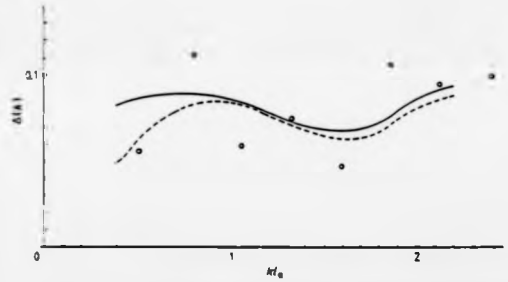


**Figure 14.** The variational estimate of the collective excitation dispersion, $\Delta(k)$, at $\nu_0 = \frac{2}{7}$ for for nine(—) and twelve(- -) particle systems. Energies are in units of $e^2/\epsilon_0 l_0$. The excitation energies of the twelve particle system (•) are shown for comparison with the SMA. These energies are defined as the total energy differences between the lowest energy states for the various L and the ground state.

(1987). They diagonalised a system with up to eight particles, using periodic boundary conditions. Their calculation of excitation energies for the system seemed to indicate the presence of a double minimum in the dispersion of the collective excitations. They also calculated the excitation spectrum using an adapted single mode approximation. They claimed that the SMA was inadequate at $\nu_0 = \frac{2}{5}$.

In figure 13 I show the results I obtained using the Feynman-Bijl ansatz together with the exact excitation energies obtained for a system of ten particles. I note that the results of the SMA seem well converged as a function of system size and are close to the exact results for the ten particle system.

I believe that my results from the SMA are close to the true result and that the SMA is indeed a good approximation at $\nu_0 = \frac{2}{5}$. I suspect that the results of Wu and Su are a spurious consequence of using periodic boundary conditions and a system with only eight particles.

**Results at $\nu_0 = \frac{2}{7}$**

In figure 14 I show our results for a system at $\nu_0 = \frac{2}{7}$. I show both the variational estimate of the dispersion, $\Delta(k)$, and the exact result for the excitation energies for a system with twelve particles.

The excitation energies for the twelve particle system seem puzzling. (See also d'Ambrumenil and Morf, 1988). It looks as if the excitations in the system move between two separate bands, giving rise to the zig-zag effect apparent in figure 14.

The variational estimate lies between the 'lower' and the 'higher' energies.

I am tempted to trust the variational estimate of the dispersion, which shows a well-defined roton minimum similar to those seen at $\nu_1 = \frac{1}{3}$ and $\nu_2 = \frac{2}{5}$. I suspect that the zig-zag effect in the excitation energies of the twelve particle system is a finite size effect. In Haldane's classification of the hierarchy of the Laughlin-like fluids the $\nu = \frac{2}{7}$ state is a second level fluid, formed from the excitations of the $\nu = \frac{2}{5}$ state, which in turn is a fluid formed from the excitations of the $\nu = \frac{1}{3}$ state. According to this classification the twelve electron system at $\nu = \frac{2}{7}$ is a Laughlin fluid with only four of the excitations of the $\nu = \frac{2}{5}$ state (d'Ambrumenil and Morf, 1988), and hence represents a very small system.

**Extensions to the N=1 Landau level**

Recently we (d'Ambrumenil and Reynolds 1988) presented evidence for the existence of a $\nu_1 = \frac{2}{7}$ Laughlin fluid even though its 'parent' state according to the hierarchical classification was unstable or only marginally stable. In fig.15 I show the variational estimate of the dispersion for collective excitations at $\nu_1 = \frac{2}{7}$ together with exact results for a system of eight particles. The variational estimate lies close to the exact results. There is also a well-defined roton minimum.

I believe that fig.15 confirms that the MacDonald and Girvin picture of the low-lying excitations is valid outside the lowest Landau level whenever the ground state is a homogeneous fluid Laughlin-like state.

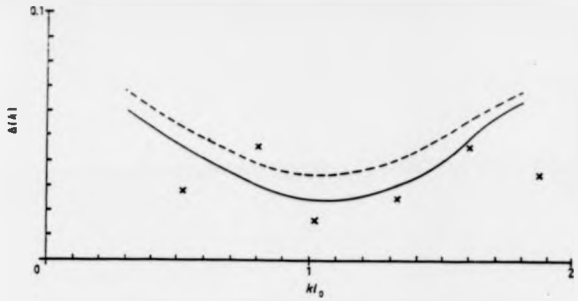


**Figure 15.** The variational estimate of the collective excitation dispersion, $\Delta(k)$, at $\nu_1 = \frac{1}{3}$ for six(- -) and eight(—) particle systems. Energies are in units of $e^2/\epsilon_0 l_0$. The excitation energies for the eight particle system (×) are shown for comparison with the SMA. These energies are defined as the total energy differences between the lowest energy states for the various L and the ground state.

## 3.4 Incoherent Single Particle Excitations: Quasiexcitons

In this section I discuss the SMA and the nature of the collective excitation spectrum at large wavevectors k.

The SMA is exact in the limit of vanishingly small wavevectors k (GMP), but starts to break down at finite wavevectors. This is because the density wave is not a sensible excitation for wavelengths smaller than the average interparticle spacing. However GMP showed that $\Delta(k)$ is still the exact first moment of the spectral function for intra-Landau level excitations. It is fruitful to consider $\Delta(k)$ in the limit of large wavevectors k because it leads to an understanding of the excitations in the FQHE.

Making use of the known exact form of the radial distribution function for FQH states (Girvin,1984) and eqns 3.10 and 3.11 it is straightforward to show that for large wavevectors k, the projected structure takes the form;

$$\overline{S_n}(k) \sim \left[L_N\left(k^2/2\right)\right]^2 (1-\nu) \exp\left(-\mid k \mid^2/2\right) \tag{3.15}$$

Inserting $\overline{S}_N(k)$ into the asymptotic form of $\Delta(k)$, gives (after some algebra), the exact first moment of the excitation spectrum in the large wavevector limit,

$$\Delta(k) = \frac{2E_{coh}(\nu)}{1-\nu} \tag{3.16}$$

Here $E_{coh}(\nu) = E(\nu) - E(1)$, is the cohesive energy per particle and $E(\nu)$ is the ground state energy per particle at a filling fraction $\nu$. This result does not rely on the validity of the SMA or on any assumption about the excited state, other than the restriction to the $N^{th}$ Landau level.

An understanding of the above result is gained by considering the projected Feynman-Bijl ansatz wavefunction, eqn 3.1, at large k. The projected density operator in eqn 3.5 acts in part like a translation operator, translating particles through a distance $k\ell_0^2$.

$$\begin{aligned} \overline{\rho}^N(k)\Phi(z_1,\ldots,z_j,\ldots,z_N) &\sim \exp\left(-ik\frac{\delta}{\delta z_j}\right)\Phi(z_1,\ldots,z_j,\ldots,z_N) \\ &\sim \Phi(z_1,\ldots,z_j+ik,\ldots,z_N) \end{aligned} \tag{3.17}$$

So that in the limit of large wavevectors $k\ell_0^2 \gg 1$, the projected Feynman-Bijl ansatz wavefunction represents a widely separated particle-hole pair. For $k\ell_0^2 \gg 1$ particles have been moved to positions where they have become uncorrelated with their surroundings. This explains the occurrence of the cohesive energy in eqn 3.16. The expectation value of the excited state's energy is large (expectation value because as $k\ell_0^2 \gg 1$ , the excited state eqn 3.1, is not a good eigenstate of the Hamiltonian). The SMA thus breaks down for large wavevectors, giving a $k^2$ (single particle) divergence in the dispersion of low lying excitations.

Kallin and Halperin 1984, suggested that the minimum energy excitations at large wavevectors, consisted of bound pairs of Laughlin quasiparticles and quasiholes. The quasiparticle being excited across the mobility gap to an unoccupied

state, leaving behind a quasihole. One expects that at a small wavevector the collective excitations cross over continuously from being density-wave like to being quasiexciton-like.

## 3.5 Summary

My findings may be summarised as follows. The dispersion for the infinite system, $\Delta(k)$, seems remarkably well converged as a function of the size of system from which I take the pair correlation function (see e.g. Fig.12). This allows me to state with some confidence that the Feynman-Bijl ansatz can be used to describe the collective excitations of any proposed Laughlin fluid even when there is no trial wavefunction readily accessible to Monte-Carlo methods.

The rapid convergence of the predicted dispersions as a function of system size may be particularly useful in the study of higher level fluids. For example I found that the variational prediction for the excitations at $\nu_0 = \frac{2}{7}$ seems typical of a system in a Laughlin-like state, whereas the exact diagonalisation of the Hamiltonian for the system of twelve particles gives a spectrum which shows large finite size effects. I am inclined to trust the variational result, because I believe that finite size effects are less pronoced in the ground state pair correlation function than in the properties of excitations of small systems.

The method also worked well in the $N = 1$ Landau level. The collective excitations predicted variationally for a system at $\nu_1 = \frac{2}{5}$ lie close to the exacty results

obtained previously for a system with eight particles (d'Ambrumenil and Reynolds, 1983). This strengthens the claim that there is indeed a Laughlin-like ground state for systems with total filling $\nu = 2 + \frac{2}{7}$ or $\nu = 2 - \frac{2}{7}$.

I demonstrated how at some small wavevector, the collective excitations might cross over continuously from being density-wave like to being quasiexciton like.

# Chapter 4

# A Generalized Cooperative Ring Exchange Theory

## 4.1 Introduction

A semiclassical treatment due to Kivelson et al 1986 (KKAS), established that the stability of the fractional quantum Hall states could be attributed to cooperative ring exchange or CRE. At flux densities corresponding to filling fractions of the lowest Landau level with odd denominators, the contributions from ring exchange processes add coherently and tend to dominate the partition function. This treatment was shown later to be consistent with Laughlin theory.

In the original version of the theory there are two regimes, one with a Wigner-crystalline ground state, and the fractional quantum Hall regime. I argue that the

theory also predicts another regime in which the Laughlin fluids are unstable and the ground state becomes compressible. This regime occurs only in higher Landau levels.

In higher Landau levels the spin-polarised Laughlin fluids are not always stable, as has already been shown on the basis of a generalization of Laughlin theory (chapters 2 and 3). I argue that this 'destabilization' of the Laughlin states emerges naturally in the ring exchange picture. Central to my argument is the observation that generalising from the lowest ($n = 0$) Landau level leads only to an increase in tunneling probabilities. At higher fillings quantum fluctuations are then strong enough to destabilize the incompressible states, and one must include contributions from events from the much larger space of (classically less favourable) configurations typical of a compressible fluid.

The remainder of this chapter is organised as follows. In the next section I outline the CRE theory of the FQHE due to KKAS. In section 4.3 I argue that the theory predicts two phase boundaries and can account for the 'destabilization' of FQH states in higher Landau levels. A summary is presented in section 4.4.

## 4.2 CRE Theory of the FQHE

In the KKAS treatment, the partition function for a system of spin-polarised two dimensional electrons in a magnetic field is expressed as a path integral (Appendix

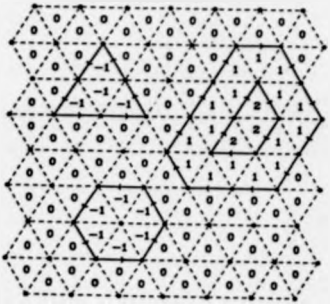

**Figure 16.** Mapping onto a spin model. The integer spin variables assigned to a plaquette reflect the net ring-exchange circulation about that plaquette in a given time slice.

C).

$$Z(\nu) = \frac{1}{N!} \sum_{p \in S_N} (-1)^p \int \prod_{j=1}^{N} D\mathbf{R}_j(\tau) \exp -S[\mathbf{R}(\tau)] \tag{4.1}$$

where the action for continuous paths is

$$S[\mathbf{R}] = \frac{1}{2} \int_0^\beta d\tau \left( -i \sum_{j=1}^{N} \left( \dot{\mathbf{R}}_j \times \mathbf{R}_j \right) \cdot \hat{\mathbf{s}} + \sum_{j \neq k} V(\mathbf{R}_j - \mathbf{R}_k) \right) \tag{4.2}$$

KKAS placed the electrons on a regular Wigner lattice (see Fig. 16) and then evaluated the partition function within the semiclassical approximation. The semiclassical approximation amounts to assuming that the classical paths (instantons), which correspond to the cooperative tunneling of rings of electrons on a Wigner lattice to permuted equivalent configurations, dominate the path integral. Quantum fluctuations are then included by expanding the action around these classical paths. KKAS then write

$$Z \approx \sum_{\alpha} D[\mathbf{R}_\alpha] \exp -S[\mathbf{R}_\alpha] \tag{4.3}$$

where the sum is over classical paths and $D[\mathbf{R}_\alpha]$ is the fluctuation determinant.

CRE processes can only preferentially stabilize the electronic ground state if there exists a factor $-1$ from the fluctuation determinant for every exchange loop. At least within the dilute instanton gas approximation, for densities of the lowest Landau level with $\nu^{-1}$ an odd integer, the fluctuation determinant does indeed contain a factor $-1$ for each exchange loop, (KKAS, 1988). I found no difficulty in generalizing this result to higher Landau levels.

The real part of the classical action is approximately proportional to the number of electrons in the ring L, that is $S_{real} \approx \alpha(\nu) L$ and the imaginary part is $\theta =$

$\pm 2\pi\,(\phi/\phi_0) + \pi\,(L-1)$ where $\phi$ is the flux enclosed by the ring and the $\pm$ refers to the positive or negative sense of rotation. The phase factor $\theta$ takes account of Fermion statistics and the Aharonov-Bohm effect arising from the coupling of the tunneling current to the magnetic field.

A general path in eqn.4.1 will contain many exchange rings and when the tunneling barrier $\alpha$ is small they are sufficently dense that they overlap and intersect, so that the action associated with such a path is not simply the sum of the actions associated with the constituent rings. A well controlled approximation for summing configurations from these paths has yet to be found. However KKAS developed an approximate method for summing contributions from an important subclass of paths by relating the sum to the partition function of the discrete Gaussian spin model (Chui and Weeks 1976). This was achieved by first dividing the time interval $\beta$ into slices of width $\tau$. Then because classical paths are exponentially localised in time, ring exchanges occuring in different time slices are essentially non interacting. Thus $Z \approx [Z_{slice}]^{\frac{\beta}{\tau}}$, where $Z_{slice}$ is the trace over all ring exchanges in a given time slice. KKAS then enumerated paths within a given time slice with a set of integer valued spins $S_\lambda$ defined as the number of clockwise minus the number of anticlockwise rings which encircle a plaquette labelled by $\lambda$, see Fig.16. The classical paths correspond to domain walls in the spin model. KKAS associated with each spin configuration an 'energy' $H_{DG}$ which is equal to the action associated with the corresponding

classical path and made the approximation $Z_{class} \approx Tr \exp(-H_{DG})$ where

$$H_{DG} = \alpha(\nu) \sum_{<\lambda,\mu>} (S_\lambda - S_\mu)^2 + i\hbar(\nu) \sum_\lambda S_\lambda \qquad (4.4)$$

Here $< \lambda, \mu >$ denotes nearest neighbour plaquettes and $\hbar(\nu) = \pi(\nu^{-1} - 1)$.

The discrete Gaussian model is known to have a phase transition at a critical value of $\alpha_c(\nu)$ (Chui and Weeks 1976). For $\alpha > \alpha_c$ the system is in a sparse domain phase (corresponding to a sparse ring phase in the electron system). KKAS associated this phase with the Wigner crystal ground state in the electron system. For $\alpha < \alpha_c$ the behaviour of the spin system is dominated by large domains (corresponding to large exchange rings in the electron system). KKAS associated this phase with the FQH states in the electron system.

Baskaran 1986 included incompressible liquid-like configurations as intermediate configurations in the treatment of KKAS. Baskaran concluded that CRE phenomena as suggested by KKAS can persist in an incompressible quantum fluid.

## 4.3 A Generalised CRE Theory of the FQHE

In this section I argue that the destabilisation of FQH states in higher Landau levels emerges naturally within the CRE picture. I claim that the two phase boundaries in the electron system, between solid and incompressible fluid (Wigner crystal and FQH state) and between incompressible and compressible fluids can be understood in terms of two Kosterlitz Thouless vortex unbinding transitions. I then introduce a general class of vortex system exhibiting two Kosterlitz Thouless unbinding

transitions and discuss how the language used to describe these systems might be adopted for the case of spin polarised electrons in a magnetic field. Finally I outline the construction of an effective Hamiltonian describing the electron case.

I begin by considering an adapted discrete Gaussian model which allows for liquid like configurations of electrons,

$$H = \alpha \sum_{<\lambda,\rho>} (S_\lambda - S_\rho)^2 + i \sum_\lambda (h + 2\pi f_\lambda) S_\lambda + \sum_{\lambda,\rho} f_\lambda \log |\mathbf{R}_\lambda - \mathbf{R}_\rho| f_\rho + \beta \sum_\lambda f_\lambda^2 \tag{4.5}$$

Here $f_\lambda$ is the derivation from the value on the regular Wigner lattice of the flux enclosed by a plaquette labelled by $\lambda$. Fluctuations in the flux enclosed by a plaquette arise from fluctuations in the area of plaquettes and so to local (charge) density fluctuations. I model these (the last two terms in equ. 4.5) by a two dimensional Coulomb gas.

For a general liquid-like configuration some plaquettes enclose excess flux quanta while others enclose fewer. Provided that plaquettes of opposite type remain closely bound together to form a 'neutral' system, a dominant fraction of ring exchange events will still interfere constructively and preferentially stabilise the ground state at densities with $\nu^{-1}$ odd. This situation arises in incompressible fluids, where global fluctuations in system size are suppressed. If the fluid becomes compressible, so that plaquettes can vary in size independently, then almost all ring exchanges interfere destructively. This argument was the basis of Baskaran 1988 result, that CRE phenomena and the consequent fractional quantisation as suggested by KKAS

can persist in an incompressible quantum fluid.

I now recast the adapted discrete Gaussian model (eqn. 4.5) into a form which will put the sparse to dense domain phase transition on the same footing as the incompressible to compressible fluid transition. I do this by integrating out the spin variables in the adapted discrete Gaussian model (eqn. 4.5) to leave a Hamiltonian describing a two dimensional system of vortices,

$$H_{VG} = \frac{\pi}{2\alpha} \sum_{\lambda,\rho} \left( \frac{h_\lambda}{2\pi} + f_\lambda \right) \log | R_\lambda - R_\rho | \left( \frac{h_\rho}{2\pi} + f_\rho \right) + \left( \frac{h_\lambda}{2\pi} + f_\lambda \right)^2 . \qquad (4.6)$$

This Hamiltonian is known to describe a system exhibiting a phase boundary. For $\alpha > \frac{\pi}{2}$ vortices of opposite vorticity are bound tightly together in pairs, and unbind in a Kosterlitz Thouless vortex unbinding transition at $\alpha = \frac{\pi}{2}$. The two phases in this vortex system may be put into correspondence with the sparse and dense ring phases in the electron system identified by KKAS.

For small values of the tunneling barrier strength $\alpha(\nu)$, one must include contributions from (classically less favourable) configurations of electrons, typical of a compressible fluid. I claim that this incompressible - compressible boundary arises from the usual form of two-dimensional melting, Kosterlitz and Thouless 1973. That is for large $\alpha$ dislocations in the electron system are bound together in pairs, and do not destroy any long range positional ordering of the electrons. Then at some small value of $\alpha$ these dislocations unbind in a Kosterlitz Thouless transition. The regime with 'free' dislocations is characteristic of a compressible fluid, as it costs only the chemical potential to add or remove dislocations and so change the size of

the system.

An intuitive physical picture for the destabilisation of FQH states now emerges. As the tunneling barrier strength $\alpha(\nu)$ is decreased one expects to encounter two phase boundaries. The first corresponds to the boundary between a solid and incompressible fluid (Wigner crystal and FQH state) at a small value of $\alpha(\nu)$. In this regime the important contributions to the partition function come from incompressible fluid-like configurations of electrons. The second boundary is between an incompressible and compressible fluid, at a smaller value of $\alpha(\nu)$. This second phase boundary can only occur in higher Landau levels, where the tunneling barrier strength $\alpha(\nu)$ is reduced. At higher fillings fluctuations are then 'strong enough' to destabilise the incompressible states. In the compressible fluid, phase coherence between large exchange rings is lost as they can no longer enclose a well defined number of flux quanta, and the consequent 'fractional quantization' cannot occur.

The introduction of dislocations into configurations of electrons results in the removal or creation plaquettes (spins). If an adapted discrete Gaussian model is to allow for configurations of electrons containing dislocations then one must introduce an additional set a variables which can take account of missing or excess spins. It seems plausible (although I could find no rigorous way of deriving it) that such an adapted discrete Gaussian model takes the form,

$$H = \alpha \sum_{<\lambda,\mu>} (S_\lambda - S_\mu - m_{\lambda\mu})^2 + 2\pi i \sum_\lambda n_\lambda S_\lambda + \log y_e \sum_\lambda m_{\lambda\nu}^2 + \log y_v \sum_\lambda n_\lambda^2 \quad (4.7)$$

Here the set of $m_{\lambda\mu}$ which are defined on the electron 'lattice', take account of dis-

locations (missing and excess spins). The set of $n_\lambda = \frac{1}{2\pi}\oint_\lambda \nabla\theta + f_\lambda$ are the 'magnetic vortices' defined on the plaquettes (dual lattice) of the electron system. The variables $y_e$ and $y_p$ have been introduced to control the size of fluctuations in $m_{\lambda_p}$ and $n_\lambda$. With dislocations being suppressed in the limit $y_e \to 0$.

The adapted discrete Gaussian model (equ.4.7), is known to describe a system exhibiting two Kosterlitz Thouless vortex unbinding transitions (Jose et al, 1978). The first is the unbinding of magnetic vortices on the dual lattice (the sparse to dense ring phase) followed by the unbinding of vortices on the electron lattice (dislocation unbinding). The three phases identified in the adapted discrete Gaussian model (equ. 4.7), may be put into correspondence with the Wigner crystal, FQH ground state and compressible fluid in the electron case.

## 4.4 Conclusions

In this chapter I argued that the destablisation of FQH states in higher Landau levels emerges naturally within the CRE picture. I claimed that the two phase boundaries in the electron system result from two consecutive Kosterlitz Thouless vortex unbinding transitions. With these vortices residing on the dual of the electron 'lattice' and the electron 'lattice' itself. I identified a general class of Hamiltonian describing such a system of vortices. I claimed that the three phases identified in the vortex gas model, might be put into correspondence with the Wigner crystal, FQH states and the compressible fluids in the electron case. I outlined how one might

derive an effective Hamiltonian within the CRE picture, which can take account of the three phases in a system of spin polarised electrons in a magnetic field.

# Chapter 5

# Summary, Open Questions and New Directions

## 5.1 Summary

The central theme of this thesis has been the existence and stability of fractional quantum Hall (FQH) states. The initial motivation for this study came from the apparent absence of a well defined FQH state at $\nu_1 = \frac{1}{3}$ in the first (n = 1) Landau level (e.g. see Willett et al 1988, Fig.6). This contrasted with the situation in the lowest Landau level, where the FQH state at $\nu_0 = \frac{1}{3}$ is found to be the most 'robust'. Further motivation came from the observation (e.g. see Clarke et al 1988) of a FQH state at $\nu_1 = \frac{2}{5}$ in the first Landau level that was better defined than its 'parent state' at $\nu_1 = \frac{1}{3}$. This result cast doubt on the hierarchical picture of the FQHE,

which predicts that the 'strength' of FQH states should decrease not increase as one proceeds down the hierarchy.

My findings may be summarised as follows. In chapter 2 I showed that for filling fractions $\nu_n$ with $\nu_n^{-1} \leq 2n+1$ the generalisations of the Laughlin state to higher ($n > 0$) Landau levels are the exact nondegenerate ground states for an electron-electron interaction that is physically unrealisable. The justification for expecting a Laughlin-like ground state at these fillings is therefore weak. I claimed that the absence of a well defined FQH state at $\nu_1 = \frac{1}{3}$ ($\nu_1^{-1} \leq 2 \times 1 + 1$) might be attributed to the ground state not being Laughlin-like at this filling.

I suggested that a hierarchical picture of the FQHE, in which a new ground state results from the condensation of a quasiparticle gas from the preceeding level, should not be taken too literally. I argued, contrary to the hierarchical picture, that the regime $5 > \nu_1^{-1} > 3$ in the first Landau level is approximately analogous to the regime $3 > \nu_0^{-1} > 1$ in the lowest Landau level, independent of whether the ground state is Laughlin-like at $\nu_1 = \frac{1}{3}$ or not. This view was supported by the results of numerical studies of collective excitations in the FQHE, presented in chapter 3.

In chapter 3 I studied the dispersion of collective excitations in the FQHE using Feynman-Bijl ansatz wavefunctions. I adpoted the approach of Girvin et al 1986, who were able to relate the dispersion to the pair correlation function. This elegant and simple method had been restricted to systems of electrons at filling fractions $\nu$ with $\nu^{-1}$ an odd integer. This was because it was only at these filling fractions

that the Laughlin wavefunction exists so that a good approximation to the pair correlation function can be found using Monte-Carlo methods. I showed that the pair correlation can be obtained from the direct diagonalizations of the Hamiltonian for a small number of electrons confined to the surface of a sphere instead of from the simulation of large systems described by the Laughlin wavefunction. I was then able to generalise the method of Girvin et al 1986 to filling fractions at which a system of electrons is expected to have a Laughlin-like incompressible ground state and for which generalisations of the Laughlin wavefunction had been suggested. The rapid convergence of the predicted dispersion as a function of system size was particularly usefully in a study of higher level fluids. I was able to investigate the dispersion in the regime $8 > \nu_1^{-1} > 3$ and add support to the claims made in chapter 2.

I found that at $\nu_1 = \frac{2}{7}$ the disperion of collective excitations was characteristic of a FQH state, with a well defined gap at $k = 0$ and a roton minimum at finite k. This contrasted with the dispersion for the parent state at $\nu_1 = \frac{1}{3}$, where I found only small gap at $k = 0$ and no clear roton miminum. I found good agreement between the exact results for the excitation energies and my variational estimate of the dispersion.

In chapter 4, I argued that a generalised cooperative ring exchange (CRE) of the FQHE predicts the occurence of two phase boundaries. The first is between a Wigner crystal and Laughlin-like ground state at small filling fractions $\nu$. The second corresponds to the boundary between a Laughlin-like ground state and a compressible

fluid ground state. This second transition occurs only in higher ($n > 0$) Landau levels. The 'destabilization' of Laughlin-like ground states emerged naturally in the CRE theory.

## 5.2 Open Questions and New Directions

In what follows I give my views on some of the open questions in the understanding of the FQHE and suggest possible new directions for research.

### Even Denominators

The observation (Willett et al, 1988) of a FQH state in a single well heterojunction at a filling $\nu_1 = \frac{1}{2}$ with an even denominator was unexpected. An equivalent quantization is not observed in the lowest Landau level at similar temperatures. Prior to this, FQH states had been exclusively observed for filling fractions with odd denominators.

The FQH state at $\nu_1 = \frac{1}{2}$ was found (Eisenstein et al, 1988) to collapse rapidly as the magnetic field is tilted away from the plane normal to the two dimensional electron gas. Since the dominant effect of tilting the magnetic field is to increase the electron spin Zeeman energy, the collapse of the $\nu_1 = \frac{1}{2}$ state suggests that this state is not spin polarised.

Haldane and Rezayi 1988, have proposed a candidiate non-polarized spin singlet wavefunction for the $\nu_1 = \frac{1}{2}$ ground state. Two features present in the experimental

regime $\nu_1 \sim \frac{1}{2}$ but not present in earlier observations FQH states in the lowest Landau level which favour their trial wavefunction are the lower magnetic field (so lower spin-Zeeman energy) and the lower correlation energy for electrons in the same cyclotron orbit.

The Haldane-Rezayi formed their trial wavefunction by putting the majority of electrons into a spin polarised $\nu_1 = \frac{1}{2}$ Laughlin state and then used the remaining flux quanta to form states into which they put the remaining electrons. The validity of the Haldane-Rezayi trial wavefunction has yet to be confirmed by quantitative numerical studies. However I believe that their wavefunction may not be a good candidiate for the $\nu_1 = \frac{1}{2}$ ground state because the $\nu_1 = \frac{1}{3}$ Laughlin state which forms its 'backbone' has only a 'marginal stability' (chapters 2 and 3). I tentatively suggest that the $\nu_1 = \frac{2}{5}$ ground state may be better backbone from which to form a $\nu_1 = \frac{1}{2}$ candidiate ground state wavefunction. This is because results of numerical studies (e.g. see chapter 3) suggest that the $\nu_1 = \frac{2}{5}$ is Laughlin-like and because there are less electrons 'left over', which must be put into energetically less favourable states. It may prove fruitful to investigate this possibility.

The mechanism by which even-denominator FQH states can be stabilised within the CRE picture is at present unclear. Making use of the dilute instanton gas approximation, Kivelson et al 1982, investigated the conditions underwhich CRE could stabilise at FQH ground state in the lowest Landau level. They found that ring exchanges of Fermions can only add in phase, when the filling fraction $\nu$ has an

odd denominator. For Bosons, even- and odd-membered rings were found to interfere with one another, and it was not apparent how CRE might stabilise the a spin singlet FQH ground state. I found no difficulty in generalising this result to higher $n > 0$ Landau levels, so that at least within the dilute instanton gas approximation it is not clear how CRE processes could stabilise the even denominator $\nu_1 = \frac{1}{2}$ ground state. The possibility for a more subtle mechanism than that considered by Kivelson et al 1988, that might allow CRE processes to stabilise even-domoninator ground states, but which is obscured by the dilute instanton gas approximation warrants investigation.

While the 'odd denominator' FQH states seem to be reasonably well understood, a clear understanding of the 'even denominator' FQH states both in the Laughlin and CRE pictures as yet to emerge.

## Spin Unpolarised Odd Denominator FQH States

The observation of an even denominator FQH state, renewed interest in the possibility of a spin unpolarised FQH ground state. Halperin 1983, was the first to point out that because of the small g factor ($g \sim 0.5$) in GaAs the electron spin degrees of freedom may play an important role in determining the FQH ground state.

While at high magnetic fields perpendicular to the plane of the two dimensional electronic structure the spin Zeeman energy will stabilise the polarised ground state, the possibility remains for a transition to an unpolarised ground state at small fields. Recently Clarke et al 1989 and Eisenstein et al 1989 reported behaviour of FQH

states consistant with such a transition. They observed that for filling fractions $\nu_0$ with $1 < \nu_0 < 2$, a transition between two distinct FQH states sharing the same odd denominator filling fraction occurs. This transition is driven by tilting the plane of the two dimensional electronic structure relative to the external magnetic field and adjusting the carrier concentration (gate voltage) to maintain constant filling. They intrepreted their results by suggesting that as the magnetic field is increased a transition from a spin unpolarised FQH state to a spin polarised one occurs.

Clarke et al 1989 also reported observations of spin unpolarised hierarchical states in the regime $1 < \nu < 2$. They claimed that consequentive higher level fluids differ in spin configuration (e.g $\frac{4}{3}$ unpolarised, $\frac{7}{5}$ partial or maximum polarised and $\frac{10}{7}$ unpolarised). They proposed that this might be because quasiparticle excitations are spin reversed, and argued as follows. Starting from the $\nu = 1 + \frac{1}{3}$, a electron ground state, with total spin $\frac{1}{2}$, the daughter state at $\nu = 1 + \frac{2}{5} = 1 + \frac{1}{2+\frac{1}{2}}$ is formed by exciting $n/2$ quasiparticles (see section 1.7). If all the quasiparticle excitations are spin reversed, then the $\nu = 1 + \frac{2}{5}$ ground state will be spin unpolarised. Extending this arguement down the hierarchy gives rise to the sequence of spin configurations suggested by Clarke et al 1989.

The observation of a transition from an unpolarised to a polarised ground state with increasing magnetic field was not unexpected but to clarify the situation more experiments along with accompanying numerically studies are needed. At the very least a re-examination of the conditions underwhich a FQH ground state may be

expected to be spin polarised is called for.

### Critical Fillings and Phase Boundaries

There is a discrepancy between the theoretical prediction for the critical filling at which Wigner crystallisation occurs and observations of FQH states. Making use of a variational wavefunction, Lam and Girvin 1984 predicted that Wigner crystallisation should occur at a filling fraction slighter larger than $\nu_0 = \frac{1}{7}$. Results from numerical studies of collective excitations (Girvin et al 1986), support this claim. They found that the energy gap separating the ground from the lowest lying excitations approaches zero at the roton minimum for the $\nu_0 = \frac{1}{7}$ FQH ground state, indicating the proximity of another phase. However these results are contrary to recent magnetotransport measurements on very low disorder GaAs/GaAlAs heterojunctions by Goldman et al 1988. They intrepreted structure in $\rho_{xy}$ near the filling fraction $\nu_0 = \frac{1}{7}$ as evidence for a developing FQH state.

Improved numerical calculations and further experimental investigations at low fillings are needed if the critical filling for Wigner crystallisation is to be determined. At the time of writing, the Wigner crystal ground state remains unobserved in semiconductor heterojunctions.

In Landau levels with index $n > 1$, the Laughlin-like ground states are predicted to be unstable for some fillings (Chapters 2 and 4). Magnetotransport measurements for this regime would be very useful in testing this prediction. Magnetotransport measurements have yet to resolve in sufficient detail structure in $\rho_{xx}$ and $\rho_{xy}$ for

Landau levels with indices $n > 1$, so as to make possible identification of any FQH states.

## A Landau-Ginzburg Theory of the FQHE

Developing an effective field theory description of the FQHE analogous to the Landau-Ginzburg theory of superconductivity is an intriguing theoretical challenge. Progress made recently in this direction (Zhang et al, 1989 and Read 1989) had has its starting point the field theory model of the FQHE, due to Girvin 1987 and Girvin and MacDonald 1987. These models which have a Chern-Simons action, exhibit most of the phenomenology of the FQHE. The partition function for these models exhibits downward cusps as a function of density and the Hall conductance is quantised at fillings $\nu = \frac{1}{m}$ with m an odd integer. Deviating from these prefered fillings, vortex solutions with finite energy, fractional charge and obeying fractional statistics can be identified with Laughlin's quasiparticles and quasiholes. Collective density fluctuations were found to be massive, although at the present level of approximation the roton minimum was not evident.

Central to a Landau-Ginzburg picture of the FQHE is the existence of an order parameter associated with some kind of symmetry breaking. The general concensus (e.g. see Girvin and MacDonald 1987) seems to be that although the usual form of off-diagonal long range order or ODLRO (Yang, 1962) seems not to exist in the FQHE ground state, there may exist a novel type of ODLRO. This algebraically decaying ODLRO is associated with the binding of the zeroes in the ground state

wavefunctions (which must necessarily be present because of the magnetic field) to the particles.

In addition to providing a deeper physical insight into the FQHE, a Landau-Ginzberg theory would allow much easier semiquantitative calculations than would a microscopic theory. A Landau-Ginzberg picture may be fruitful in an investigation of the role of disorder in the FQHE.

A Landau-Ginzberg theory may allow the FQHE to be 'unified' with existing field theories which exhibit phenomomena reminiscent of the FQHE (see e.g. Girvin and MacDonald 1983).

# Appendix A

# Single Particle Hamiltonian and Cyclotron Orbits

The Hamiltonian describing the motion a two dimensional particle with charge q and effective mass $m^*$ in a magnetic field is

$$H_0 = (\mathbf{p} - e\mathbf{A})^2 / 2m^* \tag{A.1}$$

where A is the vector potential and $\mathbf{p} = \hbar/i\nabla$.

It is often useful to write $H_0$ in so called magnetic units, by introducing the cyclotron frequency $\omega_c = eB/m^*$ and the magnetic length $l_0 = (\hbar/eB)^{\frac{1}{2}}$ and writing

$$\begin{aligned} \mathbf{r}' &= \mathbf{r}/l_0 \\ \nabla' &= l_0\nabla \\ \text{and} \quad \mathbf{A}' &= \mathbf{A}/l^2\mathbf{B} \end{aligned} \tag{A.2}$$

then

$$H_0 = \frac{\hbar\omega_c}{2}\left(\nabla'/i - \left(\frac{q}{c}\right)\mathbf{A}'\right)^2 \qquad \text{(A.3)}$$

For a particle confined to the xy plane in a perpendicular magnetic field $B_z$, the eigenfuctions of $H_0$ (the cyclotron orbits) have the energy spectrum

$$E(n) = \left(n + \frac{1}{2}\right)\hbar\omega_c \qquad \text{(A.4)}$$

where n is the Landau index.

The form of the eigenfunctions is dependent on the choice of gauge for A. In the Landau gauge $\mathbf{A} = (0, Bx, 0)$ the cyclotron orbits $\Psi$ may be written as

$$\Psi = |\, n, k\,\rangle \sim e^{iky'} H_n\left(x' - k\left(\frac{q}{c}\right)\right)\exp\left(-\left(x' - k\left(\frac{q}{c}\right)\right)^2/2\right) \qquad \text{(A.5)}$$

where the $H_n$ are Hermite polynomials. In this gauge one can easily construct an orthonormal basis which makes explicit translational invariance parallel to some direction (the x-direction say).

In the symmetric gauge $\mathbf{A} = (-By/2, Bx/2, 0)$, the cyclotron orbits are

$$\Psi = |\, n, m\,\rangle = \frac{1}{\sqrt{2\pi n!}}\left(x' + i\left(\frac{q}{c}\right)\right)^m \exp\left(-r'^2/4\right) L_n^m\left(r'^2/2\right) \qquad \text{(A.6)}$$

where $r'^2 = x'^2 + y'^2$ and the $L_n^m$ are Laguerre polynomials. In this gauge the cyclotron orbits are eigenstates of the angular momentum operator $L_z$ defined in appendix B.

One may show that the density of states per unit area in each Landau level is given by

$$\rho_0 = \frac{eB}{h} = \frac{1}{2\pi l_0^2} \qquad \text{(A.7)}$$

# Appendix B

# Conserved Angular Momentum

In the presence of a magnetic field angular momentum $J = r \times \Pi$ is not conserved, one finds that J is time dependent,

$$\begin{aligned} \frac{dJ}{dt} = r \times \frac{d\Pi}{dt} &= r \times q\left(\frac{dr}{dt} \times B\right) \\ &= q\left[(r.B)\frac{dr}{dt} - B\left(r.\frac{dr}{dt}\right)\right] \end{aligned} \tag{B.1}$$

However a conserved quantity corresponding to angular momentum is easily constructed. For a particle with charge q confined to the xy plane in a perpendicular magnetic field $B_z$, a conserved quantity $L_z$ can be defined

$$L_z = \left[\left(\frac{q}{e}\right)(r \times \Pi)_z + \frac{eBr^2}{2} + nh\right] \tag{B.2}$$

The inclusion of the Landau level index n in the definition of $L_z$ ensures that $L_z$ is positive.

$L_z$ may be written in terms of the guiding centre coordinates **R**,

$$\mathbf{R} = \mathbf{r} - \left(\frac{q}{e}\right) i_{ab}^2/\hbar\,(\mathbf{s} \times \mathbf{\Pi}) \tag{B.3}$$

which are classically just the centres of the cyclotron motion in the plane perpendicular to the magnetic field

$$L_z = (eB/2)\,R^2 - \hbar/2 \tag{B.4}$$

which in magnetic units can be written as

$$L_z/\hbar = R'^2/4 - 1/2 \tag{B.5}$$

# Appendix C

# Coherent State Path Integration

The usual procedure for constructing a path integral representation for the partition function $Z = Tre^{-\beta H}$ is as follows (see Schulman, 1981). The inverse temperature $\beta$ is divided into a large number of intervals of equal size $\epsilon$, and the resolution of the identity $P = \frac{1}{2\pi} \int d^2\mathbf{R} \mid \mathbf{R} >< \mathbf{R} \mid$ is inserted at each step.

In this appendix I sketch the essential features of this construction and discuss the nature of the paths in the resulting path integral.

In the symmetric gauge $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$ the single particle Hamiltonian $H_0$ (Appendix A) has non - orthogonal solutions (the so,called canonical coherent states), localised about the guiding centres $\mathbf{R}$,

$$< \mathbf{r} \mid \mathbf{R} >= \frac{1}{\sqrt{2\pi}l} \left( \frac{r^* - R^*}{\sqrt{2}} \right)^2 \exp\left(rR^* - r^*R\right)/4 \exp - \mid \mathbf{r} - \mathbf{R} \mid^2 /4 \qquad (C.1)$$

with

$$< \mathbf{R}', s' \mid \mathbf{R}, s >= \delta_{s,s'} \exp\left(\mathbf{R}'\mathbf{R}^* - \mathbf{R}'^*\mathbf{R}\right)/4 \exp - \mid \mathbf{R} - \mathbf{R}' \mid^2/4 \qquad \text{(C.2)}$$

and

$$P_n = \frac{1}{2\pi} \int d^2\mathbf{R} \mid \mathbf{R} >< \mathbf{R} \mid \qquad \text{(C.3)}$$

The representation of $P_n$ is not unique since the basis $\mid \mathbf{r} >$ is overcomplete (Tao, 1986) but it is a useful form for constructing path integrals.

A many body path integral may now be constructed as follows.

$$\begin{aligned} < \mathbf{R}_f \mid e^{-\beta \mathbf{H}} \mid \mathbf{R}_i > &= \int \frac{d^2\mathbf{R}_1}{2\pi}, \ldots, \int \frac{d^2\mathbf{R}_m}{2\pi} < \mathbf{R}_f \mid e^{-\epsilon \mathbf{H}} \mid \mathbf{R}_m > \ldots < \mathbf{R}_1 \mid e^{-\epsilon \mathbf{H}} \mid \mathbf{R}_i > \\ &= \int \prod_{k=1}^{m} \frac{d^2\mathbf{R}_k}{2\pi} \exp - S_m[\mathbf{R}] + 0\left(\epsilon^2\right) \end{aligned} \qquad \text{(C.4)}$$

where the discretised action is

$$\begin{aligned} S_m[\mathbf{R}] &= \sum_{n=0}^{m} \sum_{i=1}^{N} \frac{1}{4} \left( \mathbf{R}_i^{n+1} \left[ \mathbf{R}_i^{*n+1} - \mathbf{R}_i^{*n} \right] - \mathbf{R}_i^{*n} \left[ \mathbf{R}_i^{n+1} - \mathbf{R}_i^{n} \right] \right) \\ &+ \frac{1}{2} \epsilon \sum_{i \neq j}^{N} V\left( \mathbf{R}_i^{n+1}, \mathbf{R}_j^{n+1} \mid \mathbf{R}_i^{*n}, \mathbf{R}_j^{*n} \right) \end{aligned} \qquad \text{(C.5)}$$

Here $\mathbf{R}_i^n$ denotes the guiding centre of particle $i$ at (imaginary) time $t = n\epsilon$, with $\epsilon = \beta/M$.

The discrete path integral eqn.C.4 is well defined for all non-zero $\epsilon$. The continuous time limit ($\epsilon \to 0$) yields a formal path integral,

$$< \mathbf{R}_f \mid e^{-\beta \mathbf{H}} \mid \mathbf{R}_i >= \int \mathrm{D}[\mathbf{R}(\tau)] \exp - \mathrm{S}[\mathbf{R}(\tau)] \qquad \text{(C.6)}$$

where the continuous time action is

$$S[\mathbf{R}(\tau)] = \int_0^\beta d\tau \left[ \sum_{i=1}^{N} \frac{1}{4} \left( \mathbf{R}_i \dot{\mathbf{R}}_i^* - \dot{\mathbf{R}}_i \mathbf{R}_i^* \right) + \frac{1}{2} \sum_{i \neq j}^{N} V \left( \mathbf{R}_i, \mathbf{R}_j \mid \mathbf{R}_i^*, \mathbf{R}_j^* \right) \right] \quad \text{(C.7)}$$

The N-Fermion partition function

$$Z_N = \frac{1}{N!} \sum_{P \in S_N} (-1)^P \int \prod_{i=1}^{N} \frac{d^3 \mathbf{R}_i}{2\pi} < \mathbf{R}_{P(1)}, \ldots, \mathbf{R}_{P(N)} \mid \exp(-\beta \mathrm{H}) \mid \mathbf{R}_1, \ldots, \mathbf{R}_N > \quad \text{(C.8)}$$

may now be cast into the form

$$Z_N = \frac{1}{N!} \sum_{P \in S_N} (-1)^P \int D[\mathbf{R}(\tau)] \exp -S[\mathbf{R}(\tau)] \quad \text{(C.9)}$$

The non-classical paths in eqn. C.9 are continuous and differentiable nowhere (Schulman, 1981) so that inspite of offering an intuitive insight, the form adopted is only heuristic. The physical orgin of this difficulty lies in the non-orthogonality of the basis states $\mid \mathbf{R} >$. The coherent state propagator $< \mathbf{R} \mid e^{-\beta \mathrm{H}} \mid \mathbf{R}' >$ tends to the non-zero overlap $< \mathbf{R} \mid \mathbf{R}' >$ in the continuous time limit so that there is a non-zero probability for a particle to undergo an instantaneous hop over an arbitrary distance. Not withstanding these difficulties, the path integral, eqn. C.9, can be used to evaluate the partition function within the semiclassical approximation. This is because in the continuous time limit $\epsilon \to 0$ the classical paths tend to a smooth path (with possible discontinuities at the end points), so that the continuous time path integral, eqn. C.9, can be used to calculate contributions to the partition function from the classical paths, but the fluctuation determinant must be computed using the discrete time integral.

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